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# Capital accumulation and non-renewable energy resources: a special functions case

Agustin Pérez-Barahona

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# Capital accumulation and non-renewable energy resources: a special functions case

Agustin PEREZ-BARAHONA[1]

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**Abstract**

In this paper, we study the implications of assuming different technologies for physical capital accumulation and consumption. More precisely, we assume that physical capital accumulation is relatively more energy-intensive than consumption. We conclude that this hypothesis, together with the possibility of technical progress (in particular, energy-saving technical progress), has important implications on economic growth. This model entails some technical difficulties. However, we provide a full analytical characterization of both short and long-run dynamics using Gauss Hypergeometric functions.

**Keywords**: non-renewable resources, energy-saving technical progress, special functions.

**JEL classification**: C68, O30, O41, Q30, Q43

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[1] CORE, Université catholique de Louvain, Belgium. E-mail: perez@core.ucl.ac.be

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# 1 Introduction

A central problem in Environmental Economics is the production of energy by means of non-renewable energy resources, such as fossil fuels. Indeed, authors as Dasgupta and Heal (1974), Hartwick (1989), and Smulders and Nooij (2003) point out that the usage of non-renewable energy resources implies a limit to the economic growth of modern economies.

The standard literature on non-renewable energy resources (see for instance, Dasgupta and Heal (1974, 1979), Solow (1974a,b), and Stiglitz (1974)) gives central position to physical capital accumulation to offset the constraint on production possibilities due to non-renewable energy resources. Indeed, as Stigliz (1974) and Dasgupta and Heal (1979) claim, physical capital accumulation can compensate the negative effect of non-renewable energy resources on economic growth, even without technical progress (the economy can achieve, at least, steady state). However, this literature assumes the same technology for both physical capital accumulation and consumption. Nevertheless, since physical capital is a crucial element to solve the problem imposed by non-renewable energy resources, the assumptions on the technology for physical capital accumulation play an important role too. Indeed, the assumption of same technology for both physical capital accumulation and consumption implies (among other things) that the energy intensity of both sectors is the same. However, data does not to support this implication. There are not available data to construct an energy-intensity measurement of the theoretical physical capital and consumption sectors considered in this kind of literature. However, we can use databases at sector level as proxy for our energy-intensity data. Physical capital accumulation, *i.e.*, equipment good production, usually involves the transformation of raw material such as iron, steel, non-ferrous metals or non-metallic minerals. Moreover, transport and storage also plays important role on these kinds of activities. However, consumption good sectors are more related with other activities, such as food, clothes or construction. Azomahou *et al.* (2006), taking the Structural Analysis Database of the OECD and the energy databases of the IEA, build an energy intensity measurement (ratio between energy consumption and value added) of 14 sectors of the economy. They find that this ratio (mean) is particularly high for iron and steel (0.809) and transport and storage (0.85). Moreover, non-ferrous metals (0.599) and non-metallic minerals (0.507) also present considerably high energy intensity. However, the activities more related with consumption goods present lower energy intensity. Indeed, food and tobacco accounts for an energy intensity of 0.134. Furthermore, the energy intensity of textile and leather (0.082) and construction (0.018) is lower. Then, we can conclude that data seems to support that physical capital accumulation is more energy-intensive than consumption.

In this paper, we study the implications of assuming different technologies for physical capital accumulation and consumption. More precisely, we assume that physical capital accumulation is relatively more energy-intensive than consumption. We use a general equilibrium model with three sectors (non-durable good, physical capital and extraction sector), based on Dasgupta and Heal (1974), Hartwick (1989) and Pérez-Barahona (2006c). We point out four main contributions of this paper. First, in contrast with the previously presented claim of Stiglitz (1974) and Dasgupta and Heal (1979), technical progress plays a crucial role to sustain long-run growth in our paper. Indeed, Pérez-Barahona (2006c), assuming that physical capital accumulation is more energy-intensive than consumption, and total capital depreciation (*i.e.*, physical capital as a flow variable), shows that the economy decreases at constant rate (with all the endogenous variables converging to zero asymptotically) if there is no growth of technical progress. In our paper, considering physical capital as a stock variable, we show that neither balanced growth path (BGP) nor steady state equilibrium do not exist if there is no growth of technical progress.

The second contribution of this paper is mainly methodological. This model entails some technical difficulties. In particular, we have to deal with an optimal control problem with mixed constraints and two states variables: capital accumulation and non-renewable energy resource stock. Moreover, the capital accumulation law is a non-linear function with respect to investment. However, we provide a close-form solution of the optimal solution paths of our variables in levels for every time $t$ by applying the technique of Special Functions representation. This is very well known method in the field of mathematical physics to describe the behavior of dynamical systems. This tool provides a full analytical characterization of both short and long-run dynamics, without using qualitative techniques, such as phase diagrams. In this paper, we use a family of Special Functions called Gauss Hypergeometric functions (see Boucekkine and Ruiz-Tamarit (2004) for an application to the Lucas-Uzawa model).

Applying the previously presented technique of Special Functions representation, we provide the necessary and sufficient conditions for existence and uniqueness of BGP, which is the third contribution of this paper. Moreover, we prove that the economy asymptotically converges to the BGP. This result contrast with Pérez-Barahona (2006c), where the economy does not present dynamical transition. The reason is that Pérez-Barahona (2006) assumes physical capital as a flow variable. Here, we eliminate that simplification considering physical capital as a durable good.

Finally, since the Special Functions representation provides a close-form solution of the optimal solution paths of our variables in levels, the fourth contribution of our paper is that we can explicitly study the monotonicity properties of the optimal trajectories in levels, which is typically unallowed in standard approaches. We obtained that the assumption of physical capital as relatively more energy-intensive sector than consumption, and the technical progress (in particular, energy-saving technical progress), introduce new results of non-monotonicity of the optimal trajectories in levels, which contrast with Dasgupta and Heal (1974)'s results about the monotonicity of consumption and extraction of non-renewable energy resources.

In this paper, we give special attention to energy-saving technologies. The idea behind energy-saving technical progress is energy efficiency, which is define as the inverse of the energy intensity (quantity of energy per unit of output). The importance of this kind of technologies is pointed out in Boucekkine and Pommeret (2004). The interest for energy efficiency is shared by developed countries, such as the EU, regarding to the Environment and Energy Policy. In particular, the European Commission, through the Green Paper "Doing more with less" (COM (2005)), opened the debate about the opportunities of energy-efficiency savings in the EU. Indeed, according to COM (2005), the European Union (EU) could save at least 20% of its present energy consumption improving energy-efficiency.

The paper is organized as follows. Section 2 describes our economy, providing the dynamical system corresponding to the optimal solution. In Section 3, we study the BGP equilibrium. Section 4 solves the dynamical system presented in Section 2 using Gauss Hypergeometric functions. In Section 5, we provide the analytical expressions of the optimal solution paths. Section 6 studies the monotonicity properties of the optimal trajectories in levels. Finally, some concluding remarks are considered in Section 7.

# 2 The model

Let us consider a three sector economy with exhaustible energy resources, where the population is constant[1], based on Dasgupta and Heal (1974), Hartwick (1989) and Pérez-Barahona (2006c). The final good sector only produces a non-durable good, which can be assigned to consumption or investment. The final good is produced by means of AK technology, where the physical capital (durable good) is the only input. The durable good sector accumulates physical capital by means of a technology defined over two inputs: investment and energy. Finally, the extraction sector directly produces energy by extracting non-renewable energy resources from a given stock.

## 2.1 Final good sector

The final good sector produces a non-durable good $Y(t)$ by means of the following $AK$ technology:

$$Y(t) = A(t)K(t), \tag{1}$$

where $A(t)$ is the disembodied technical progress (or scale parameter), and $K(t)$ represents the physical capital, which is the only input to produce the final good. Here, we consider AK technology because it is the simplest set-up[2]. Final good is used either to consume, $C(t)$, or to invest in physical capital, $I(t)$, verifying the budget constraint of the economy:

$$Y(t) = C(t) + I(t). \tag{2}$$

The standard literature on non-renewable energy resources (see for instance Dasgupta and Heal (1974, 1979), Stiglitz (1974), Solow (1974), Hartwick (1989), or Smulders and Nooij (2003)) assumes the same technology for physical capital accumulation, $\dot{K}(t)$, and consumption[3]. However, as we will see in Section 2.2, our paper assumes that physical capital accumulation is a function of investment and non-renewable energy resources, $R(t)$, (*i.e.*, $\dot{K}(t) = g[I(t), R(t)]$). This implies that physical capital accumulation is relatively more energy-intensive than consumption.

[1]As Pérez-Barahona (2006c) observes, Stigliz (1974) studies an economy with non-renewable resources and exponential growing labor force.

[2]Nevertheless, one could include more inputs such as energy (for final good production), or labor.

[3]They consider $Y(t) = F[K(t), R(t)]$, where $Y(t) = C(t) + I(t)$ and $\dot{K}(t) = I(t)$.

## 2.2 Durable good sector

The durable good sector accumulates physical capital, $\dot{K}(t)$, by using two inputs. On the one hand, the durable good sector takes the fraction of final good devoted to investment. And on the other, this sector uses the energy, $R(t)$, produced by the extraction sector. The technology for physical capital accumulation is represented by the following Cobb-Douglas function:

$$\dot{K}(t) = [\theta(t)R(t)]^{\alpha} I(t)^{1-\alpha}, \text{ with } 0 < \alpha < 0, \tag{3}$$

where $\theta(t)$ denotes the embodied energy-saving technical progress, and $K(0)$ is given. In this paper, technical progress is considered as an exogenous variable.

Two observations should be made at this point. First, notice that we assume substitutability between energy and investment. Indeed, there is a very well known debate about substitutability *vs.* complementarity. If one considers the idea of minimum energy requiring to use a machine, the assumption of complementarity should be chosen. However, following the argument of Dasgupta (1979), one can assume substitutability if $I(t)$ is interpreted as final good service. Following that, we assume that the provision of a flow of final good $I(t)$ implies the provision of a certain energy flow.

As second observation, one can notice that the main different between Pérez-Barahona (2006c) and our model is that Pérez-Barahona (2006c) considers physical capital as a flow variable instead of a stock. As this author points out, this is a simplification which assumes total capital depreciation for each period. However, our model eliminates that simplification, considering physical capital as a stock variable. The main consequence of our assumption is the following. The model of Pérez-Barahona (2006c) has not dynamical transition, *i.e.*, the economy is always in the BGP. However, as we will prove later, considering physical capital as a stock generates dynamical transition towards the BGP. Moreover, despite of having the same long-run growth rate as Pérez-Barahona (2006c), our BGP levels are different because they are affected by the growth rates of technical progress.

## 2.3 Extraction sector

Energy is directly produced by extracting non-renewable energy resources, $R(t)$, from a given homogeneous stock $S(t)$. As Dasgupta and Heal (1974), and Hartwick (1989), we assume costless extraction[4]. The law of motion of

[4]Dasgupta and Heal (1974): "Extraction cost do not appear to introduce any great problem, provided that we assume any non-convexities". Indeed, one can easily introduce

the stock of non-renewable energy resources is described by the expression

$$R(t) = -\dot{S}(t), \text{ where } S(0) \text{ is given.} \tag{4}$$

Since one can not extract more than the available stock, the following restriction should be included

$$S(t) \geq 0. \tag{5}$$

## 2.4 Optimal solution

The central planner (optimal solution) maximizes the instantaneous utility function of the representative household:

$$\max W = \int_0^\infty ln[C(t)]exp(-\rho t)dt, \text{ with } \rho > 0,$$

subject to the equations (1)-(5), where $\rho$ is the time preference parameter (it is assumed to be a positive discount factor).

One can easily rewrite the problem as optimal control problem with mixed constraints:

$$\max \int_0^\infty ln[A(t)K(t) - I(t)]exp(-\rho t)dt, \text{ with } \rho > 0$$

subject to:

$$\dot{K}(t) = [\theta(t)R(t)]^\alpha I(t)^{1-\alpha}, \text{ with } 0 < \alpha < 0,$$
$$R(t) = -\dot{S}(t),$$
$$S(t) \geq 0,$$
$$\text{with } K(0) \text{ and } S(0) \text{ given,}$$

where $K(t)$ and $S(t)$ are the state variables, and $I(t)$ and $R(t)$ are the control variables. Following Sydsæter *et al.* (1999), the sufficient condition for this problem is a dynamical system described by the next proposition:

**Proposition 1.** *Given the initial conditions $K(0)$ and $S(0)$, the optimal solution of our optimal control problem is a path $\{C(t), R(t), I(t), S(t), K(t)\}$ that satisfies the following conditions:*

$$\dot{K}(t) = \theta(t)^\alpha R(t)^\alpha I(t)^{1-\alpha}, \tag{6}$$

extraction cost $EC(t)$ by modifying the budget constraint of the economy:

$$Y(t) = C(t) + I(t) + EC(R, S), \text{ where } \partial EC/\partial R > 0 \text{ and } \partial EC/\partial S \leq 0.$$

$$\frac{\dot{C}(t)}{C(t)} = (1-\alpha)A(t)\theta(t)^{\alpha}\left(\frac{R(t)}{I(t)}\right)^{\alpha} - \alpha\left(\frac{\dot{\theta}(t)}{\theta(t)} - \frac{\dot{I}(t)}{I(t)} + \frac{\dot{R}(t)}{R(t)}\right) - \rho, \tag{7}$$

$$(1-\alpha)A(t)\theta(t)^{\alpha}\left(\frac{R(t)}{I(t)}\right)^{\alpha} = \alpha\frac{\dot{\theta}(t)}{\theta(t)} + (1-\alpha)\left(\frac{\dot{I}(t)}{I(t)} - \frac{\dot{R}(t)}{R(t)}\right), \tag{8}$$

$$A(t)K(t) = C(t) + I(t), \tag{9}$$

$$\dot{S}(t) = -R(t). \tag{10}$$

The proof is provided in Appendix A ■

This system has five equations and five unknowns [5]. Equation (6) is the accumulation law of physical capital. Notice that, in this model, energy is an input to accumulate physical capital. If one denotes $\dot{K}(t) = g[\theta(t), I(t), R(t)]$, for a general utility function $U[C(t)]$, equation (7) could be rewritten as

$$\frac{U_{cc}\dot{C}}{U_c} - \rho = \frac{\dot{g}_I}{g_I} - Ag_I.$$

This is the Ramsey optimal savings relation. Similarly, equation (8) is equivalent to the following expression:

$$A(t)g_I = \frac{\dot{g}_R}{g_R},$$

which is the efficient condition for using up the non-renewable resource, *i.e.*, a variant of the Hotelling rule. Finally, equation (9) is the budget constraint of the economy, and equation (10) is the law of motion of the stock of non-renewable energy resources.

# 3 Balanced growth path equilibrium

Let us define balanced growth path equilibrium (BGP) as the situation where all the endogenous variables grow at constant rate, *i.e.*, $x(t) = \overline{x} \cdot exp(\gamma_x t)$.

[5] Indeed, there are two more equations: one transversality condition (TC) for each state variable:

$$\lim_{t\to\infty} \psi(t)K(t) = 0 \text{ and} \lim_{t\to\infty} \lambda(t)S(t) = 0,$$

where $\psi(t)$ and $\lambda(t)$ are the corresponding lagrangian multipliers. We will use these two TC conditions in order to obtain the two constants involved in the calculation of the endogenous variables of the dynamical system. Actually, one could write a system of seven equations (the five previous equations and these two TC) and seven unknowns (the preceding unknowns and the two constants related with the state variables).

Following Solow (1974a), we assume that $A(t) = \overline{A} \cdot exp(\gamma_A t)$ and $\theta(t) = \overline{\theta} \cdot exp(\gamma_\theta t)$ for all $t$, where $\overline{A}, \overline{\theta}, \gamma_A, \gamma_\theta$ are strictly positive and exogenous parameters. Indeed, we will prove later that both kind of (exogenous) technical progress should grow at a high enough rate in order to have BGP with positive growth rates.

The following proposition summarizes the growth rates of the endogenous variables of our model, along the BGP (necessary condition):

**Proposition 2.** *Along the BGP, $Y(t)$, $C(t)$, and $I(t)$ grow at rate*

$$\gamma_Y = \gamma_C = \gamma_I = \gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho,$$

*the growth rate of the stock of physical capital is*

$$\gamma_K = \gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - \rho,$$

*and extraction, $R(t)$, and stock of non-renewable energy resources, $S(t)$, grow at rate*

$$\gamma_R = \gamma_S = -\rho.$$

The proof is provided in Appendix B ■

Parameter $\rho$ is the (positive) discount rate, which represents the degree of impatience of the household. Indeed, the greater $\rho$ the more impatient the household. Then, household prefer a higher level of present consumption than to postpone it. As one can observe from the previous proposition, the greater $\rho$ the greater decreasing rate of the stock of non-renewable energy resources. This is because, since household does not want to postpone consumption, he prefers to extract a higher amount of energy resource in order to have greater present consumption.

Furthermore, since the stock of non-renewable energy resources is lower, this reduces the long-run growth rates of the economy for a given growth rate of technical progress. However, this negative effect of impatience could be compensated by greater growth rates of both disembodied technical progress ($\gamma_A$) and energy-saving technical progress ($\gamma_\theta$). Indeed, from Proposition 2, if $(1-\alpha)\gamma_A + \alpha\gamma_\theta > \alpha\rho$, then $\gamma_Y (= \gamma_C = \gamma_I) > 0$ and $\gamma_K > 0$. The reason is that technical progress (in particular, energy-saving technical progress) reduces the energy intensity of the economy (*i.e.,* increases energy efficiency). If one defines energy intensity as the ratio $R(t)/Y(t)$, it is easy to prove that, in our model, the growth rate of energy intensity along the BGP is[6]

$$\gamma_{R/Y} = -\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A\right),$$

which is negative for $\gamma_A > 0$ and $\gamma_\theta > 0$. This is consistent with empirical works, such as Azomatou *et al.* (2004).

However, the following observation should be done regarding to the cases less or equal. If $(1-\alpha)\gamma_A + \alpha\gamma_\theta = \alpha\rho$, then $\gamma_Y (= \gamma_C = \gamma_I) = \gamma_A$ and $\gamma_K = 0$. An interesting example of this case is $\gamma_A = 0$ and $\gamma_\theta = \rho$, which implies (see Proposition 2) $\gamma_Y (= \gamma_C = \gamma_I) = 0$ and $\gamma_K = 0$, *i.e.*, steady state equilibrium[7]. This result contrasts with Dasgupta and Heal (1974). As we observe in the Introduction, these authors achieve steady state equilibrium without technical progress. In our case, where physical capital accumulation is relatively more energy-intensive than consumption, we need technical progress (in particular, energy-saving technical progress) to reach, at least, steady state equilibrium. Indeed, let us consider the case $(1-\alpha)\gamma_A + \alpha\gamma_\theta < \alpha\rho$. If $\gamma_A > 0$ and/or $\gamma_\theta > 0$, then $\gamma_K < 0$. However, this is not possible because we do not consider physical capital depreciation. A particular example of our last case is $\gamma_A = \gamma_\theta = 0$. Rewriting the dynamical system (6)-(10), one can easily obtain that, along the BGP (if it exists), all variables decrease at the constant rate $\rho$. However, as for the previous case, it is not possible that the stock of physical capital decreases because we do not consider physical capital depretation. Then, BGP (neither steady state) does not exist if there is no growth of technical progress.

# 4 Analytical solution using Gauss Hypergeometric Functions

In Section 4, we will fully characterize the dynamics of our economy. As we explain in Section 1, the equilibrium of our economy is described by the dynamical system (6)-(10). Here, we will use a new technique in Economics called Special Functions[8]. This is a very well known technique in the field

[6]Notice that we can consider two alternatives definitions of energy intensity, $R(t)/K(t)$ or $R(t)/I(t)$. For these definitions, the corresponding BGP growth rates are $\gamma_{R/K} = -\left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A\right)$ and $\gamma_{R/I} = -\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A\right)$.

[7]As one will verify later, our analysis is also valid for $\gamma_A = 0$ or $\gamma_\theta = 0$ (we do not consider negative growth rates of technical progress). The case $\gamma_A = \gamma_\theta = 0$ is discussed at the end of this paragraph.

[8]See Abramowitz and Stegun (1970).

of Mathematical Physics. The tool of Special Functions provides us a full analytical characterization of the dynamics of our system, without using qualitative techniques such as phase diagrams. However, this methodology has not been sufficiently explored in Economic Theory and, in particular, in Growth models[9].

Since we are interested in the dynamics towards the BGP, it is useful to detrend our dynamical system[10]. Let us define the following change of variable $\tilde{x}(t) = x(t)/exp(\gamma_x t)$. Then, the transformed (detrended) variables will be constant along the BGP (if it exists). Applying this change of variable, one obtains the following detrended dynamical system with five equations and five unknowns $(\tilde{K}, \tilde{C}, \tilde{R}, \tilde{I}, S)$:

$$\dot{\tilde{K}}(t) + \left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - \rho\right)\tilde{K}(t) = \overline{\theta}^\alpha \tilde{R}(t)^\alpha \tilde{I}(t)^{1-\alpha}, \tag{11}$$

$$\frac{\dot{\tilde{C}}(t)}{\tilde{C}(t)} = (1-\alpha)\overline{A\theta}^\alpha \left(\frac{\tilde{R}(t)}{\tilde{I}(t)}\right)^\alpha - \alpha\left(\frac{\dot{\tilde{R}}(t)}{\tilde{R}(t)} - \frac{\dot{\tilde{I}}(t)}{\tilde{I}(t)}\right) - \gamma_\theta - \frac{1-\alpha}{\alpha}\gamma_A, \tag{12}$$

$$(1-\alpha)\overline{A\theta}^\alpha \left(\frac{\tilde{R}(t)}{\tilde{I}(t)}\right)^\alpha = (1-\alpha)\left(\frac{\dot{\tilde{I}}(t)}{\tilde{I}(t)} - \frac{\dot{\tilde{R}}(t)}{\tilde{R}(t)}\right) + \gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A, \tag{13}$$

$$\tilde{C}(t) + \tilde{I}(t) = \overline{A}\tilde{K}(t), \tag{14}$$

$$\dot{S}(t) = \tilde{R}(t)exp(-\rho t). \tag{15}$$

Achieving the analytical solution of this dynamical system, one fully characterizes the dynamics of the economy described by in this paper. Moreover, if there is BGP, the variables involved in this (detrended) dynamical system will be constant along the BGP. Furthermore, since $\gamma_A$ and $\gamma_\theta$ are exogenous parameters, one can easily recovers the original variables by retrieving the change of variable.

In order to obtain the analytical solution of the dynamical system (11)-(15), we will apply the following strategy:

**Step 1.** Let us define the ratio $X(t) = \frac{\tilde{R}(t)}{\tilde{I}(t)}$.

**Step 2.** One can easily observes that equation (13) is a Bernoulli's equation (see Section 4.2). Then, we can obtain the analytical solution for the ratio $X(t)$.

[9]See Boucekkine and Ruiz-Tamarit (2004) for an application to the Lucas-Uzawa model.

[10]Notice that this is not a compulsory step to apply Special Functions. However, eliminating the long-run trends allows us to have less messy expressions.

**Step 3.** Taking $X(t)$ into equation (12), one obtains a linear first-order differential equation. Solving this differential equation, the analytical expression for $\tilde{C}(t)$ is straightforward.

**Step 4.** Rewriting equation (11), one gets

$$\dot{\tilde{K}}(t) + \left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - r\right)\tilde{K}(t) = \overline{\theta}^\alpha X(t)^\alpha[\overline{A}\tilde{K}(t) - \tilde{C}(t)].$$

Since we have the analytical solution for $\tilde{C}(t)$ (Step 3), the previous expression is a linear first-order differential equation. By solving it, one gets the analytical solution for $\tilde{K}(t)$. Notice that, this step will involve the application of Special Functions. In particular, we will use a family of Special Functions called Gauss Hypergeometric functions. Later, we will provide a briefly introduction to these fuctions.

**Step 5.** From equation (14),

$$\tilde{I}(t) = \overline{A}\tilde{K}(t) - \tilde{C}(t).$$

Since we know both $\tilde{C}(t)$ (Step 3) and $\tilde{K}(t)$ (Step 4), the previous expression gives us the analytical solution for $\tilde{I}(t)$.

**Step 6.** Since we know $\tilde{I}(t)$ and $X(t)$, one can easly gets $\tilde{R}(t)$.

**Step 7.** Finally, from equation (15), $S(t)$ is obtained by solving a linear first-order differential equation.

## 4.1 Step 1.

Let us define the ratio $X(t) = \frac{\tilde{R}(t)}{\tilde{I}(t)}$. Taking logs and differentiating with respect to $t$, one gets

$$\frac{\dot{X}(t)}{X(t)} = \frac{\dot{\tilde{R}}(t)}{\tilde{R}(t)} - \frac{\dot{\tilde{I}}(t)}{\tilde{I}(t)}. \tag{16}$$

## 4.2 Step 2.

Applying equation (16) into equation (13), one finds

$$\dot{X}(t) = \frac{\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A}{1-\alpha}X(t) + \overline{A\theta}^\alpha X(t)^{1+\alpha}. \tag{17}$$

One can observe that equation (17) is a Bernoulli's equation. Following Sydsæter *et al.* (1999), a Bernoulli's equation is defined as

$$\dot{x}(t) = q(t)x(t) + r(t)x(t)^n,$$

which has the following solution:

$$x(t) = exp\left(\frac{p(t)}{1-n}\right)\left(\varphi + (1-n)\int r(t)exp(-p(t))dt\right)^{\frac{1}{1-n}},$$

where $p(t) = (1-n)\int q(t)dt$, $\varphi$ is a constant, and $n \neq 1$.

Then, the solution of the Bernouilli's equation (17) is

$$X(t) = exp\left(\frac{1}{\alpha}Zt\right)\left(\varphi + \frac{\alpha\overline{A\theta}^{\alpha}}{Z}exp(Zt)\right)^{-\frac{1}{\alpha}}, \tag{18}$$

where $Z = \frac{\alpha}{1-\alpha}\left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A\right)$, and $\varphi$ is a constant which should be calculated.

## 4.3 Step 3.

Applying equations (16) and (18) into equation (12), one finds that

$$\frac{\dot{\tilde{C}}(t)}{\tilde{C}(t)} = \frac{\overline{A\theta}^{\alpha}exp(Zt)}{\varphi + \alpha\overline{A\theta}^{\alpha}exp(Zt)} - \frac{Z}{\alpha}, \tag{19}$$

which is a linear first-order differential equation.

A linear first-order differential equation is defined as

$$\dot{x}(t) + a(t)x(t) = b(t),$$

whose solution is given by the expression (Sydsæter *et al.* (1999))

$$x(t) = x(0)exp\left(-\int_0^t a(\xi)d\xi\right) + \int_0^t b(\tau)exp\left(-\int_\tau^t a(\xi)d\xi\right)d\tau.$$

Applying this result into equation (19), we find that

$$\tilde{C}(t) = \tilde{C}(0)exp\left[\frac{1}{\alpha}\left(ln(\varphi Z + \alpha\overline{A\theta}^{\alpha}exp(Zt)) - ln(\varphi Z + \alpha\overline{A\theta}^{\alpha}) - Zt\right)\right], \tag{20}$$

where $\tilde{C}(0)(= C(0))$ is the initial condition for consumption, which should be determined.

## 4.4 Step 4.

Rewriting equation (11), one gets

$$\dot{\tilde{K}}(t) + \left(\frac{1-\alpha}{\alpha}Z - \rho - \overline{A\theta}^{\alpha}X(t)^{\alpha}\right)\tilde{K}(t) = -\overline{\theta}^{\alpha}X(t)^{\alpha}\tilde{C}(t) \tag{21}$$

Since from the previous steps we know the analytical solution for $X(t)$ (equation (18)) and $\tilde{C}(t)$ (equation (20)), $\tilde{K}(t)$ is the solution of the linear first-order differential equation (21).

Applying the previous result about the solution of a linear first-order differential equation, from equation (21) one finds that

$$\tilde{K}(t) = exp\left[-\frac{1}{\alpha}ln(\varphi Z + \alpha\overline{A\theta}^{\alpha})\right]$$

$$\cdot exp\left\{\frac{1}{\alpha}\left[ln(\varphi Z + \alpha\overline{A\theta}^{\alpha}exp(Zt)) - ((1-\alpha)Z - \alpha\rho)t\right]\right\}$$

$$\cdot\left\{K(0) + \frac{Z}{Z+\rho}\frac{\tilde{C}(0)}{\alpha\overline{A}}\left[\frac{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}exp(Zt)}\right)}{exp((Z+\rho)t)} - {}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)\right]\right\}, \quad (22)$$

where

$${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}exp(Zt)}\right) = {}_2F_1\left(1, \frac{Z+\rho}{Z}, 1+\frac{Z+\rho}{Z}; -\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}exp(Zt)}\right),$$

$${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) = {}_2F_1\left(1, \frac{Z+\rho}{Z}, 1+\frac{Z+\rho}{Z}; -\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right),$$

and $K(0)(=\tilde{K}(0))$ is the (given) initial condition for capital.

As one can observe, equation (22) involves Special Functions (${}_2F_1$). In this paper, we use a family of Special Functions called Gauss Hypergeometric functions[11], which are defined as

$${}_pF_q(a;b;z) = \sum_{n=0}^{\infty}\frac{(a_1)_n \times (a_2)_n \times \ldots (a_p)_n}{(b_1)_n \times (b_2)_n \times \ldots (b_q)_n}\frac{z^n}{n!},$$

where $a = (a_1, a_2, \ldots, a_p)$, $b = (b_1, b_2, \ldots, b_q)$, and $(x)_n$ is the Pochhammer symbol, defined by:

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)},$$

where $\Gamma(x)$ is the *Gamma function.*

The proof of equation (22) is provided in Appendix C ■

[11]See Boucekkine and Ruiz-Tamarit (2004) for a quick overview of Gauss Hypergeometric functions.

## 4.5 Step 5.

Since we have the solution of $\tilde{C}(t)$(equation (20)) and $\tilde{K}(t)$(equation (22)), from equation (14) one finds that

$$\tilde{I}(t) = exp\left\{\frac{1}{\alpha}\left[ln(\varphi Z + \alpha\overline{A\theta}^{\alpha}exp(zt)) - ln(\varphi Z + \alpha\overline{A\theta}^{\alpha}) - Zt\right]\right\}$$

$$\cdot\left\{\overline{A}exp((Z+\rho)t)\zeta(t) - \tilde{C}(0)\right\}, \tag{23}$$

where

$$\zeta(t) =$$

$$\left\{K(0) + \frac{Z}{Z+\rho}\frac{\tilde{C}(0)}{\alpha\overline{A}}\left[\frac{1}{exp((Z+\rho)t)}{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}exp(Zt)}\right) - {}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)\right]\right\}.$$

## 4.6 Step 6.

Since $X(t)$ is defined as $X(t) = \tilde{R}(t)/\tilde{I}(t)$, taking the solution of $X(t)$ (equation (18)) and $\tilde{I}(t)$ (equation (23)), one finds the solution for the energy flow $\tilde{R}(t)$:

$$\tilde{R}(t) = exp\left[-\frac{1}{\alpha}ln(\varphi Z + \alpha\overline{A\theta}^{\alpha})\right]Z^{\frac{1}{\alpha}}exp\left(\frac{1}{\alpha}Zt\right)$$

$$\cdot\left\{\overline{A}exp\left\{\left(\frac{1-\alpha}{\alpha}Z - \rho\right)t\right\}\zeta(t) - \tilde{C}(0)exp\left(-\frac{1}{\alpha}Zt\right)\right\}. \tag{24}$$

## 4.7 Step 7.

Since we have the solution for $\tilde{R}(t)$ (equation (24)), one can find the solution for the stock of non-renewable energy resources, $S(t)$, by solving equation (15), which is a Linear first-order differential equation. The solution of this equation is

$$S(t) = S(0) + \int_0^t \tilde{R}(\tau)exp(-\rho\tau)d\tau, \tag{25}$$

where $S(0)$ is given. The corresponding integral is equal to

$$\int_0^t \tilde{R}(\tau)exp(-\rho\tau)d\tau = [1] + [2] + [3] + [4],$$

where

$$[1] = \frac{(\varphi Z + \alpha \overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} \overline{A} K(0)}{2(\frac{1}{\alpha}Z - \rho) - Z} \left\{ exp\left[ \left( 2\left( \frac{1}{\alpha}Z - \rho \right) - Z \right) t \right] - 1 \right\},$$

$$[2] = A_2 \frac{1}{Z + \Omega} \frac{Z}{Z + \rho}$$

$$\cdot \left[ {}_3F_2 \left( a; b; -\frac{\varphi Z}{\alpha \overline{A}\overline{\theta}^{\alpha}} \right) - exp\left( -(Z + \Omega)\frac{Z + \rho}{Z} t \right) {}_3F_2 \left( a; b; -\frac{\varphi Z}{\alpha \overline{A}\overline{\theta}^{\alpha} exp(Zt)} \right) \right],$$

$$[3] = -A_2 \frac{{}_2F_1 \left( -\frac{\varphi Z}{\alpha \overline{A}\overline{\theta}^{\alpha}} \right)}{2(\frac{1}{\alpha}Z - \rho) - Z} \left\{ exp\left[ \left( 2\left( \frac{1}{\alpha}Z - \rho \right) - Z \right) t \right] - 1 \right\},$$

$$[4] = \frac{(\varphi Z + \alpha \overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} C(0)}{\rho} \left( exp(-\rho t) - 1 \right),$$

with

$$A_2 = (\varphi Z + \alpha \overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}} \frac{Z^{\frac{1+\alpha}{\alpha}}}{Z + \rho} \frac{C(0)}{\alpha},$$

$$\Omega = \left( 2\rho - \frac{2 - \alpha}{\alpha} Z \right) \left( \frac{Z}{Z + \rho} \right),$$

$$a = \left( 1, \frac{Z + \rho}{Z}, \frac{Z + \rho}{Z} \frac{Z + \Omega}{Z} \right),$$

$$b = \left( 1 + \frac{Z + \rho}{Z}, 1 + \frac{Z + \rho}{Z} \frac{Z + \Omega}{Z} \right),$$

and

$$\left| -\frac{\varphi Z}{\alpha \overline{A}\overline{\theta}^{\alpha}} \right| < 1.$$

The proof is provided in Appendix D ■

## 4.8 Computation of the constants $\varphi$ and $C(0)$

Equations (20), and (22)-(25) are the analytical solutions for, respectively, $\tilde{C}(t)$, $\tilde{K}(t)$, $\tilde{I}(t)$, $\tilde{R}(t)$, and $S(t)$. However, we still need to determine the value of the constants $\varphi$ and $C(0)(=\tilde{C}(0))$, in order to finish the calculations. To do this, we use the two transversality conditions (TC) involved by the states variables of our model:

$$\lim_{t \to \infty} \psi(t) K(t) = 0, \tag{26}$$

$$\lim_{t \to \infty} \lambda(t) S(t) = 0, \tag{27}$$

where $\psi(t)$ and $\lambda(t)$ are the corresponding Lagrangian multipliers. In addition, the TCs will also imply restrictions on the parameters, which allow us to complete the full characterization of the optimal trajectories.

### 4.8.1 TC for the stock of physical capital ($K(t)$)

From equation (A.1) (see Appendix A), we obtain the shadow price of physical capital, $\psi(t)$:

$$\psi(t) = \frac{(\varphi Z + \alpha \overline{A}\overline{\theta}^{\alpha})^{\alpha}}{(1-\alpha)\overline{\theta}^{\alpha} C(0)} (\varphi Z + \alpha \overline{A}\overline{\theta}^{\alpha} exp(Zt))^{-\frac{1-\alpha}{\alpha}}. \tag{28}$$

Taking equation (22), one gets[12]

$$\psi(t)K(t) = \frac{(\varphi Z + \alpha \overline{A}\overline{\theta}^{\alpha} exp(Zt))}{(1-\alpha)\overline{\theta}^{\alpha} C(0)}$$

$$\cdot \left\{ K(0) + \frac{Z}{Z+\rho} \frac{C(0)}{\alpha \overline{A}} \left[ exp(-(Z+\rho)t)_2F_1 \left( \frac{-\varphi Z}{\alpha \overline{A}\overline{\theta}^{\alpha} exp(Zt)} \right) - {}_2F_1 \left( \frac{-\varphi Z}{\alpha \overline{A}\overline{\theta}^{\alpha}} \right) \right] \right\}.$$

Then, the TC for $K(t)$ (equation (26)) implies that

$$C(0) = \frac{\alpha \overline{A}}{{}_2F_1 \left( -\frac{\varphi Z}{\alpha \overline{A}\overline{\theta}^{\alpha}} \right)} \frac{Z+\rho}{Z} K(0), \tag{29}$$

where $K(0)$ is given.

Equation (29) establishes a first relationship between $\varphi$ and $C(0)$. In order to have a system of two equations and two unknowns, we can obtain the second equation from the TC for the non-renewable energy resources (equation (27)).

### 4.8.2 TC for the stock of non-renewable energy resource ($S(t)$)

Since $\lambda(t) = \lambda$ for all $t$ (see Appendix A)[13], equation (27) implies that

$$\lim_{t \to \infty} S(t) = 0. \tag{30}$$

The transversality condition for $S(t)$ implies that the stock of non-renewable energy resources can not infinitely increase (*i.e.*, as in Boucekkine and Ruiz-Tamarit (2004), non-explosive solution) and it should be depleted in the infinite (this is because we assume that this kind of energy resources are essential and there are not alternative resources, such as renewable energy resources).

[12]Notice that $K(t) = \tilde{K}(t) exp \left[ \left( \frac{1-\alpha}{\alpha} Z - \rho \right) t \right]$.

[13]Since $q(t) = 0$, $\lambda$ is easily obtained taking equation (28) into equation (A.2) in Appendix A.

Taking equation (29) into equation (25), we get that

$$S(t) = S(0) - \frac{(\varphi Z + \alpha \overline{A\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} C(0)}{\rho} (exp(-\rho t) - 1) - A_2 \frac{1}{Z+\Omega} \frac{Z}{Z+\rho}$$

$$\cdot \left[ {}_3F_2 \left( -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}} \right) - exp\left( -(Z+\Omega) \frac{Z+\rho}{Z} t \right) {}_3F_2 \left( -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} exp(Zt)} \right) \right], \quad (31)$$

where

$$ {}_3F_2 \left( -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}} \right) = {}_3F_2 \left( a; b; -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}} \right),$$

$$ {}_3F_2 \left( -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} exp(Zt)} \right) = {}_3F_2 \left( a; b; -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} exp(Zt)} \right).$$

From equation (31), if the term $(\Omega + Z)\frac{Z+\rho}{Z}$ is negative[14], then $S(t)$ will infinitely increases. Hence, $(\Omega + Z)\frac{Z+\rho}{Z}$ should be greater than zero in order to avoid explosive solutions. Since $\rho$ and $Z$ are positive, we have to ensure that $\Omega + Z$ is positive too[15]. Then, from the definition of $\Omega$ (equation (25)), we can establish the following proposition:

**Proposition 3.** *Any particular non-explosive solution to the dynamical system (6)-(10) has to satisfy the following condition:*

$$\rho > \frac{2}{3} \left( \gamma_\theta + \frac{1-\alpha}{\alpha} \gamma_A \right).$$

Moreover, Proposition 2 establishes that the growth rate of technical progress should be high enough to ensure BGP equilibrium. Indeed, we can consider both conditions in the following proposition:

**Proposition 4.** *Any particular BGP solution to the dynamical system (6)-(10) has to satisfy the following condition:*

$$\frac{2}{3} \left( \gamma_\theta + \frac{1-\alpha}{\alpha} \gamma_A \right) < \rho < \left( \gamma_\theta + \frac{1-\alpha}{\alpha} \gamma_A \right). \quad (32)$$

The interpretation of Proposition 4 is the following. As we observe in Section 3, $\rho$ represents the household's impatience degree. Indeed, the greater is $\rho$

[14] ${}_3F_2(a; b; 0) = 1$ for $a$ and $b$ different than zero.

[15] Notice that, from Proposition 2 and the definition of $\Omega$ (equation (25)), $\Omega$ should be less or equal to 0 in order to have BGP.

the less important is future for the household (*i.e.*, the more impatient is the household). Proposition 2 establishes that technical progress should be high enough ($(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A) > \rho$) to guarantee BGP, which is the right hand side (RHS) of condition (32). However, the left hand side (LHS) of condition (32) establishes that the growth rate of technical progress should be not too high ($\frac{2}{3}(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A) < \rho$). If this is not the case, an explosive solution will appear (see Proposition 3). Nevertheless, an explosive solution is not possible because of the transversality condition for $S(t)$.

After eliminating the possibility of explosive solution, we can establish a condition on the parameters to deplete the stock of non-renewable energy resources in the infinite. From equation (31) and condition (32), it is clear that

$$\lim_{t\to\infty} S(t) =$$
$$S(0) + \frac{C(0)}{\rho}\left(\frac{Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}} - A_2 \frac{1}{Z+\Omega}\frac{Z}{\Omega+\rho} {}_3F_2\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right). \quad (33)$$

From equation (30), $\lim_{t\to\infty} S(t) = 0$. Then, we obtain the following condition replacing $A_2$ in equation (33) and equalizing to zero:

$$S(0) =$$
$$C(0)\left(\frac{Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}} \left[\left(\frac{Z}{Z+\rho}\right)^2 \frac{1}{\alpha}\frac{1}{Z+\Omega} {}_3F_2\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) - \frac{1}{\rho}\right], \quad (34)$$

where $S(0)$ is given.

Equation (34) is the second relationship between $\varphi$ and $C(0)$. Then, equations (29) and (34) is a non-linear system of two equations and two unknowns. Solving this system, $\varphi$ and $C(0)$ will be determined.

# 5 The optimal solution paths

Section 4 provided the analytical solution of the dynamical system (11)-(15) using Gauss Hypergeometric Functions. But this is the solution of the detrended dynamical system. However, one can easily recover the solution of the original dynamical system (6)-(10) by multiplying each detrended variable by its corresponding BGP growth rate, *i.e*, $x(t) = \tilde{x}(t) \cdot exp(\gamma_x t)$. Moreover, Section 4 also established the parameters constraints to have BGP solution. In Section 5, we put together all the previous conditions, providing the optimal solution paths of our economy.

## 5.1 Computation of the constant $\varphi$

Taking equation (29) into equation (34), we reduce the previous non-linear system of two equations ((29) and (34)) and two unknowns ($\varphi$ and $C(0)$) to the following single equation for $\varphi$, where $C(0)$ is directly found (equation (29)) once $\varphi$ is determined:

$$\frac{S(0)}{K(0)} = \alpha\overline{A}\left(\frac{Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}$$
$$\cdot\left[\frac{Z}{Z+\rho}\frac{1}{\alpha}\frac{1}{Z+R}\frac{{}_3F_2\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)} - \frac{Z+\rho}{Z}\frac{1}{\rho}\frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}\right], \tag{35}$$

where $K(0)$ and $S(0)$ are given.

Since equation (25) requires $\left|-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right| < 1$, we only consider parameterizations with solution for equation (35) such that[16]

$$\varphi \in \left(-\frac{\alpha\overline{A\theta}^{\alpha}}{Z}, \frac{\alpha\overline{A\theta}^{\alpha}}{Z}\right). \tag{36}$$

## 5.2 Optimal solution paths for the ratio $X(t)$

From equation (18), rearranging terms, we get the following proposition for the optimal solution path for $X(t)$:

**Proposition 5.** *Under the conditions (32) and (36), if equation (35) admits a unique solution, then*

**(i)** *it does exist a unique path for $X(t)$, starting from $X(0) = \left(\frac{Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}$, such that*

$$X(t) = \left(\frac{Z}{\varphi Z exp(-Zt) + \alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}; \tag{37}$$

**(ii)** *this equilibrium path shows transitional dynamics, approaching asymptotically to the constant*

$$X_{BGP} = \left(\frac{Z}{\alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}.$$

[16]If some particular parametrization generates a solution for the equation (35) out the interval (36), we should study the continuation formulas of the Gauss Hypergeometric representation for $S(t)$. However, this is beyond of the objective of this paper.

Moreover, it is easy to see that condition (36) implies that the starting point, the transitional dynamics, and the BGP of $X(t)$ are always positive:

**Corollary 1.** *Under the conditions (32) and (36), the optimal solution path for $X(t)$ is positive.*

### 5.3 Optimal solution paths for the consumption $C(t)$

From equation (20), we recover $C(t)$ by multiplying $\tilde{C}(t)$ by $exp(\gamma_C t)$ (see Proposition 2). Then, rearranging terms, we establish the following proposition:

**Proposition 6.** *Under the conditions (32) and (36), if equation (35) admits a unique solution, then*

**(i)** *it does exist a unique path for $C(t)$, starting from*

$$C(0) = \frac{\alpha\overline{A}}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}\frac{Z+\rho}{Z}K(0),$$

*such that*

$$C(t) = C(0)\left[\left(\frac{\varphi Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right)exp(-Zt) + \frac{\alpha\overline{A\theta}^{\alpha}}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right]^{\frac{1}{\alpha}}$$

$$\cdot exp\left\{\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho\right)t\right\}; \tag{38}$$

**(ii)** *this equilibrium path shows transitional dynamics, approaching asymptotically to the unique BGP*

$$C_{BGP}(t) = C(0)\left(\frac{\alpha\overline{A\theta}^{\alpha}}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}} exp\left\{\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho\right)t\right\};$$

*where $K(0)$ is given.*

Regarding to the sigh of the optimal solution path, for $C(0) > 0$, condition (36) implies positive starting point, transitional dynamics and BGP. From Proposition 6, the condition for $C(0) > 0$ is that ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) > 0$.

Then, we can establish the following corollary:

**Corollary 2.** *Under the conditions (32), (36), and* ${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right) > 0$, *the optimal solution path for* $C(t)$ *is positive.*

## 5.4 Optimal solution paths for the physical capital $K(t)$

Taking equation (29) into equation (22) and rearranging terms , we obtain the optimal solution path for the detrended physical capital. Then, multiplying $\tilde{K}(t)$ by $exp(\gamma_K t)$, one can establish the Proposition 7:

**Proposition 7.** *Under the conditions (32) and (36), if equation (35) admits a unique solution, then*

**(i)** *it does exist a unique path for* $K(t)$, *starting from* $K(0)$, *such that*

$$K(t) = K(0)(\varphi Z + \alpha \overline{A\theta}^{\alpha})^{-\frac{1}{\alpha}} \frac{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} exp(Zt)}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right)} \left(\frac{\varphi Z}{exp(Zt)} + \alpha \overline{A\theta}^{\alpha}\right)^{\frac{1}{\alpha}}$$

$$\cdot exp\left\{\left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - \rho\right) t\right\} \qquad (39);$$

**(ii)** *this equilibrium path shows transitional dynamics, approaching asymptotically to the unique BGP*

$$K_{BGP}(t) =$$

$$K(0)\left(\frac{\alpha \overline{A\theta}^{\alpha}}{\varphi Z + \alpha \overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}} \frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right)} exp\left\{\left(\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - \rho\right) t\right\};$$

*where* $K(0)$ *is given.*

Furthermore, as in the previous case, we can establish the following corollary for positive optimal solution path:

**Corollary 3.** *Under the conditions (32), (36), and* ${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right) > 0$, *the optimal solution path for* $K(t)$ *is positive, for a given* $K(0) > 0$.

The proof is provided in Appendix E ■

## 5.5 Optimal solution paths for the investment $I(t)$

Applying equation (29) into equation (23), one gets the optimal solution path for the detrended investment. After multiplying $\tilde{I}(t)$ by $exp(\gamma_I t)$ and rearranging terms, we find the following proposition:

**Proposition 8.** *Under the conditions (32) and (36), if equation (35) admits a unique solution, then*

**(i)** *it does exist a unique path for $I(t)$, starting from*

$$I(0) = \overline{A}K(0)\left(1 - \alpha\frac{Z+\rho}{Z}\frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}\right),$$

*such that*

$$I(t) = \frac{\overline{A}K(0)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}\left(\frac{\varphi Z + \alpha\overline{A\theta}^{\alpha}exp(Zt)}{(\varphi Z + \alpha\overline{A\theta}^{\alpha})exp(Zt)}\right)^{\frac{1}{\alpha}}$$

$$\cdot\left[{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}exp(Zt)}\right) - \alpha\frac{Z+\rho}{Z}\right]exp\left\{\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho\right)t\right\}; \quad (40)$$

**(ii)** *this equilibrium path shows transitional dynamics, approaching asymptotically to the unique BGP*

$$I_{BGP}(t) = \frac{\overline{A}K(0)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}\left(\frac{\alpha\overline{A\theta}^{\alpha}}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}\left(1 - \frac{Z+\rho}{Z}\alpha\right)$$

$$.exp\left\{\left(\gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho\right)t\right\};$$

*where $K(0)$ is given.*

It is easy to prove that condition (32) implies $1 > \alpha\frac{Z+\rho}{Z}$. If ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) > 1$, then $I(0) > 0$. Since ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}exp(Zt)}\right) > {}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) > 1$ (see Appendix E), equation (40) will be positive too. Then, we establish the following corollary:

**Corollary 4.** *Under the conditions (32), (36), and ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) > 1$, the optimal solution path for $I(t)$ is positive, for a given $K(0) > 0$.*

## 5.6 Optimal solution paths for the output $Y(t)$

Since $Y(t) = C(t) + I(t)$, Propositions 6 and 8 provides the optimal solution path for the output:

**Proposition 9.** *Under the conditions (32) and (36), if equation (35) admits a unique solution, then*

**(i)** *it does exist a unique path for $Y(t)$, starting from $Y(0) = \overline{A}K(0)$, such that*

$$Y(t) = \overline{A}K(0) \left( \frac{\varphi Z + \alpha \overline{A\theta}^{\alpha} exp(Zt)}{(\varphi Z + \alpha \overline{A\theta}^{\alpha}) exp(Zt)} \right)^{\frac{1}{\alpha}} \frac{{}_2F_1 \left( -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} exp(Zt)} \right)}{{}_2F_1 \left( -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}} \right)}$$

$$\cdot exp \left\{ \left( \gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho \right) t \right\}; \tag{41}$$

**(ii)** *this equilibrium path shows transitional dynamics, approaching asymptotically to the unique BGP*

$$Y_{BGP}(t) = Y(t) = \overline{A}K(0) \left( \frac{\alpha \overline{A\theta}^{\alpha}}{\varphi Z + \alpha \overline{A\theta}^{\alpha}} \right)^{\frac{1}{\alpha}} \frac{1}{{}_2F_1 \left( -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}} \right)}$$

$$\cdot exp \left\{ \left( \gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho \right) t \right\};$$

*where $K(0)$ is given.*

Concerning to the sign of the optimal solution path, we proceed as in Corollary 3:

**Corollary 5.** *Under the conditions (32), (36), and ${}_2F_1 \left( -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}} \right) > 0$, the optimal solution path for $Y(t)$ is positive, for a given $K(0) > 0$.*

## 5.7 Optimal solution paths for the extraction of non-renewable resources $R(t)$

Since $\tilde{R}(t) = X(t)\tilde{I}(t)$, we get the optimal solution path for the detrended non-renewable resources from Proposition 5 and equations (23) and (29) (notice that $\tilde{C}(0) = C(0)$). Considering $\gamma_R = -\rho$, we recover the optimal solution path for $R(t)$. Then, we establish the following proposition:

**Proposition 10.** *Under the conditions (32) and (36), if equation (35) admits a unique solution, then*

**(i)** *it does exist a unique path for $R(t)$, starting from*

$$R(0) = \overline{A}K(0) \left( \frac{Z}{\varphi Z + \alpha \overline{A\theta}^{\alpha}} \right)^{\frac{1}{\alpha}} \left( 1 - \alpha \frac{Z+\rho}{Z} \frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right)} \right),$$

*such that*

$$R(t) = \left( \frac{Z}{\varphi Z + \alpha \overline{A\theta}^{\alpha}} \right)^{\frac{1}{\alpha}} \frac{\overline{A}K(0)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right)} \left[ {}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} exp(Zt)}\right) - \alpha \frac{Z+\rho}{Z} \right] \cdot exp(-\rho t); \quad (42)$$

**(ii)** *this equilibrium path shows transitional dynamics, approaching asymptotically to the unique BGP*

$$R_{BGP}(t) = \left( \frac{Z}{\varphi Z + \alpha \overline{A\theta}^{\alpha}} \right)^{\frac{1}{\alpha}} \frac{\overline{A}K(0)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right)} \left[ 1 - \alpha \frac{Z+\rho}{Z} \right] \cdot exp(-\rho t);$$

*where $K(0)$ is given.*

Following the same reasoning as in Corollary 4, we can establish the following condition for positive optimal solution path for $R(t)$:

**Corollary 6.** *Under the conditions (32), (36), and ${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right) > 1$, the optimal solution path for $R(t)$ is positive, for a given $K(0) > 0$.*

## 5.8 Optimal solution paths for the stock of non-renewable resources $S(t)$

Taking equation (31) into equation (25), rearranging terms, we get the optimal solution path for the stock of non-renewable energy resources:

**Proposition 11.** *Under the conditions (32) and (36), if equation (35) admits a unique solution, then*

**(i)** *it does exist a unique path for $S(t)$, starting from $S(0)$, such that*

$$S(t) = C(0)\left(\frac{Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}$$

$$\cdot\left\{\frac{\left(\frac{Z}{Z+\rho}\right)^{2}\frac{1}{\alpha}\frac{1}{Z+\Omega}}{exp\left((Z+\Omega)\frac{Z+\rho}{Z}t\right)}{}_3F_2\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}exp(Zt)}\right)-\frac{1}{\rho}\frac{1}{exp(\rho t)}\right\}$$

$$\cdot exp(-\rho t); \tag{43}$$

**(ii)** *this equilibrium path shows transitional dynamics, approaching asymptotically to zero;*

*where $K(0)$ and $S(0)$ are given.*

Regarding to the sigh of the optimal solution path for the stock of non-renewable resources, for $C(0) > 0$, it is enough to require that $exp(\rho t)$ increases faster than $exp\left((Z+\Omega)\frac{Z+\rho}{Z}t\right)$:

**Corollary 7.** *Under the conditions (32), (36), ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) > 0$, and $\rho > (Z+\Omega)\frac{Z+\rho}{Z}$, the optimal solution path for $S(t)$ is positive, for $K(0) > 0$ and $S(0) > 0$ given*[17].

Finally, we conclude this section with a corollary of all the conditions to guarantee positive optimal solution paths of our economy:

**Corollary 8.** *Under the conditions (32), (36), ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) > 1$, $\rho > (Z+\Omega)\frac{Z+\rho}{Z}$, ${}_3F_2\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) > 1$ and , the optimal solution paths for $X(t)$, $C(t)$, $K(t)$, $I(t)$, $Y(t)$, $R(t)$, and $S(t)$, are positive, for $K(0) > 0$ and $S(0) > 0$ given.*

[17]Notice that $(Z+\Omega)\frac{Z+\rho}{Z} > 0$ (see Section 4.8.2). Moreover, the condition ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right) > 0$ is required to guarantee $C(0) > 0$.

# 6 On the monotonicity of the optimal solution paths

Applying Gauss Hypergeometric representation, Section 5 provided the closed-form solution of the optimal paths of our variables in level. An important outcome of these closed-form solutions is that we can explicitly study the monotonicity of the optimal trajectories of the variables in level. These kinds of studies are typically unallowed in standard approaches[18].

## 6.1 The relationship between $\varphi$ and $S(0)/K(0)$

We will see later that the sign of $\varphi$ is important to determine the monotonicity properties of our optimal solutions paths. Therefore, we have to study the sign of $\varphi$ before dealing with the monotonicity issue. From condition (35), taking all parameters as given, one observes that the ratio $S(0)/K(0)$ decides $\varphi$ (and, in particular, the sign of $\varphi$)[19]. Let us define $(S(0)/K(0))^*$ as the ratio corresponding to $\varphi = 0$. Then,

$$\left(\frac{S(0)}{K(0)}\right)^* = \alpha\overline{A}\left(\frac{Z}{\alpha\overline{A}\theta^\alpha}\right)^{\frac{1}{\alpha}}\left(\frac{Z}{Z+\rho}\frac{1}{\alpha}\frac{1}{Z+\Omega} - \frac{Z+\rho}{Z}\frac{1}{\rho}\right). \tag{44}$$

From (35), applying the implicit function theorem, one obtains that

$$\frac{\partial\varphi}{\partial(S(0)/K(0))} < 0.$$

The proof is provided in Appendix F ■

Hence, for $S(0)/K(0) > (<)(S(0)/K(0))^*$, $\varphi$ is negative (positive).

## 6.2 Monotonicity of the ratio $X(t)$

From equation (37), we easily obtains that

$$\frac{dX(t)}{dt} = \varphi\left[\frac{Z}{\alpha}\frac{X(t)}{\varphi + \frac{\alpha\overline{A}\theta^\alpha}{Z}exp(Zt)}\right]$$

Taking Corollary 8, one can observe that the term between the square brackets is positive. Then, the sign of the derivative is determined by the sign of

[18] An exception is Dasgupta and Heal (1974), where only consumption and extraction can be analyzed in level.

[19] Notice that Corollary 8 implies that both left and right hand side of (35) are positive.

$\varphi$. If $\varphi < (>)0$, then $dX(t)/dt < (>)0$ for all $t$. Moreover, if $\varphi = 0$, then $X(t)$ is constant for all $t$. Since the sign of $\varphi$ is determined by the ratio $S(0)/K(0)$ (see Section 6.1), we can establish the following proposition:

**Proposition 12.**

**(1)** *If $S(0)/K(0) > (<)(S(0)/K(0))^*$, then $X(t)$ decreases (increases) monotonically for all $t$.*

**(2)** *If $S(0)/K(0) = (S(0)/K(0))^*$, then*

$$X(t) = \left(\frac{Z}{\alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}$$

*for all $t$.*

See Appendix H, Figure 1, for an illustration.

## 6.3 Monotonicity of consumption

Equation (38) is the optimal solution path for consumption. First, let us study the detrended consumption

$$\tilde{C}(t) = C(0)\left[\left(\frac{\varphi Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right) exp(-Zt) + \frac{\alpha\overline{A\theta}^{\alpha}}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right]^{\frac{1}{\alpha}}. \tag{45}$$

From equation (45), we get

$$\frac{d\tilde{C}(t)}{dt} =$$

$$-\varphi\left\{\tilde{C}(t)\frac{Zexp(-Zt)}{\alpha}\left(\frac{\varphi Zexp(-Zt) + \alpha\overline{A\theta}^{\alpha}}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}\right)^{-1}\frac{1}{\varphi + \frac{\alpha\overline{A\theta}^{\alpha}}{Z}}\right\} \tag{46}$$

As for $X(t)$, the expression between braces is positive. Then, the sign of $\varphi$ determines the sign of the derivative. Hence, we can establish the following proposition:

**Proposition 13.**

**(1)** *If $S(0)/K(0) > (<)(S(0)/K(0))^*$, then $\tilde{C}(t)$ increases (decreases) monotonically for all $t$.*

**(2)** *If* $S(0)/K(0) = (S(0)/K(0))^*$*, then*

$$\tilde{C}(t) = \alpha\overline{A}\frac{Z+\rho}{Z}K(0)$$

*for all* $t$.

See Appendix H, Figure 2, for an illustration.

Knowing the monotonicity properties of $\tilde{C}(t)$, let us study $C(t)$. As we established before, $C(t) = \tilde{C}(t) \cdot exp(\gamma_C t)$, where $\gamma_C = \gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho$. Then, it is straightforward that

$$\frac{dC(t)}{dt} = exp(\gamma_C t)\left(\frac{d\tilde{C}(t)}{dt} + \tilde{C}(t)\gamma_C\right). \tag{47}$$

Taking Corollary 8, one can observe that all terms in equation (47) are positive, except $d\tilde{C}(t)/dt$. However, we can study this derivative from the monotonicity properties of $\tilde{C}(t)$. If $S(0)/K(0) > (S(0)/K(0))^*$, then $d\tilde{C}(t)/dt > 0$ for all $t$. Consequently, from equation (47), $dC(t)/dt > 0$ for all $t$. If $S(0)/K(0) = (S(0)/K(0))^*$, then $d\tilde{C}(t)/dt = 0$ for all $t$. Nevertheless, from equation (47), we conclude that $dC(t)/dt > 0$ for all $t$.

However, the difficult case arises for $S(0)/K(0) < (S(0)/K(0))^*$ because $d\tilde{C}(t)/dt < 0$ for all $t$. Replacing equation (46) into equation (47), and rearranging terms, it yields

$$\frac{dC(t)}{dt} = exp(\gamma_C t)\tilde{C}(t)\left[\gamma_C - \left(\frac{Z}{\alpha}\frac{\varphi Z}{\varphi Z + \alpha\overline{A\theta}^\alpha exp(Zt)}\right)\right]. \tag{48}$$

As in the previous section, all terms of equation (48) are positive. Since $exp(Zt)$ increases with time, the term between brackets achieves its maximum value when $t = 0$. Thus, if

$$\gamma_C \geq \frac{Z}{\alpha}\frac{\varphi Z}{\varphi Z + \alpha\overline{A\theta}^\alpha}(\equiv \gamma_C^*),$$

then, $dC(t)/dt > 0$ for all $t > 0$[20]. However, if $\gamma_C < \gamma_C^*$, there exist a strictly positive $t^*$ such that $dC(t)/dt < 0$ for all $0 < t < t^*$, $dC(t)/dt = 0$ for $t = t^*$, and $dC(t)/dt > 0$ for all $t > t^*$. It is easy to prove that

$$t^* = \frac{ln\left[\frac{\varphi Z}{\alpha\overline{A\theta}^\alpha}\left(\frac{Z-\alpha\gamma_C}{\alpha\gamma_C}\right)\right]}{Z}.$$

[20]If $\gamma_C = \gamma_C^*$, then $dC(t)/dt = 0$ for $t = 0$, and $dC(t)/dt > 0$ for all $t > 0$.

The proof is provided in Appendix G ■

Moreover, it is easy to see that

$$\frac{dt^*}{d\rho} = \frac{1}{\gamma_C(Z - \alpha\gamma_C)}.$$

From Proposition 2, taking the definition of $Z$ (equation (18)), one concludes that $dt^*/d\rho > 0$. This result implies that the more impatient is the household (*i.e.,* the greater is $\rho$) the greater is the number of periods where consumption decreases $(t^*)$[21].

We summarize the monotonicity properties of $C(t)$ in the following proposition:

**Proposition 14.**

**(1)** *If $S(0)/K(0) \geq (S(0)/K(0))^*$, then $C(t)$ increases monotonically for all $t$.*

**(2)** *If $S(0)/K(0) < (S(0)/K(0))^*$, then*

- **(2.1)** *if $\gamma_C \geq \gamma_C^*$ then $C(t)$ increases monotonically for all $t > 0$;*
- **(2.2)** *if $\gamma_C < \gamma_C^*$ then $C(t)$ decreases monotonically for all $0 < t < t^*$, and increases monotonically for all $t > t^*$.*

See Appendix H, Figure 3, for an illustration. Moreover, Figure 4 considers an example of initially decreasing consumption, but after $t^* = 99.85$ periods, consumption increases monotonically.

The economic interpretation of Propositions 13 and 14 is the following. If the proportion of non-renewable energy resources with respect to physical capital is high enough (*i.e.*, $S(0)/K(0) > (S(0)/K(0))^*$), then the detrended consumption ($\tilde{C}(t)$) increases monotonically until its BGP level. Moreover, this effect is reinforced by the growth rates of technical progress through the long-run growth rate of consumption. Therefore, the consumption ($C(t)$) increases monotonically for all $t > 0$. If $S(0)/K(0) = (S(0)/K(0))^*$, the detrended consumption is constant (its BGP level) for all $t > 0$. However, as in the previous case, technical progress allows for a consumption increasing monotonically for all $t > 0$. Finally, if the proportion of non-renewable

[21]Unfortunately, we can not provide a general conclusion of the effect of technical progress on $t^*$.

energy resources is too low (*i.e.*, $S(0)/K(0) < (S(0)/K(0))^*$), the detrended consumption decreases monotonically for all $t > 0$. However, if the growth rates of technical progress are high enough to achieve a high enough long-run growth rate of consumption (*i.e.*, $\gamma_c \geq \gamma_c^*$) , then the consumption increases monotonically for all $t > 0$. Nevertheless, if the growth rates of technical progress are not so high (*i.e.*, $\gamma_c < \gamma_c^*$), then the consumption decreases monotonically until $t^*$. And then, consumption increases monotonically for all $t > t^*$ because the technical progress effect is now high enough (notice that technical progress increases with time).

An observation should be made at this moment. Taking a standard model of non-renewable energy resources with same technology for both physical capital and consumption, Dasgupta and Heal (1974) also described a non-monotonic behavior. However, we find two main differences between Dasgupta and Heal's result and our case. First, notice that what matters for Dasgupta and Heal's result is the size of both $S(0)$ and $K(0)$. However, since our model assumes that physical capital accumulation is more energy intensive that consumption, the key element for our result is the proportion of $S(0)$ with respect to $K(0)$. Second, according to Dasgupta and Heal (1974), if the initial stocks of $S(0)$ and $K(0)$ are high enough, the consumption initially rises and, after some periods, it will decreases. However, in our model, the non-monotonic behavior appears when the proportion of $S(0)$ with respect to $K(0)$ is not high enough. Indeed, as we explained before, if the growth rates of technical progress are high enough ($\gamma_c \geq \gamma_c^*$) the negative effect of a low proportion of $S(0)$ with respect to $K(0)$ is compensated. Otherwise, if the growth rates of technical progress are not so high ($\gamma_c < \gamma_c^*$), the consumption decreases during an initial period. But, since technical progress rises with time, consumption increases after some periods.

## 6.4 Monotonicity of physical capital, investment and output

Equation (39) provides the optimal solution path for $K(t)$. Let us study the detrended capital:

$$\tilde{K}(t) = K(0)(\varphi Z + \alpha \overline{A\theta}^{\alpha})^{\frac{1}{\alpha}} \frac{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} exp(Zt)}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right)} \left(\frac{\varphi Z}{exp(Zt) + \alpha \overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}. \quad (49)$$

From equation (49), taking the formula (E.1) from Appendix E, we get

$$\frac{d\tilde{K}(t)}{dt} = \tilde{K}(t)\frac{\varphi Z^2}{\alpha exp(Zt)}$$

$$\cdot\left[\frac{Z+\rho}{2Z+\rho}\frac{1}{\overline{A}\theta^{\alpha}}\frac{{}_2F_1\left(a,b,c;-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}exp(Zt)}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}exp(Zt)}\right)} - \frac{1}{\alpha\overline{A}\theta^{\alpha}+\varphi Z exp(Zt)}\right], \tag{50}$$

where

$$a = 2; b = 1 + \frac{Z+\rho}{Z}; c = b + 1.$$

Since ${}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}exp(Zt)}\right) > 0$, applying the Euler integral representation as in Appendix D, one verifies that ${}_2F_1\left(a,b,c;-\frac{\varphi Z}{\alpha\overline{A}\theta^{\alpha}exp(Zt)}\right) > 0$. Therefore, following Corollary 8, all terms in equation (50) are positive. However, we can neither establish the sign of $d\tilde{K}(t)/dt$ nor conditions on the parameters to find out that sign. Nevertheless, we can conclude that $\tilde{K}(t)$ is not necessarily monotonic. Moreover, since $K(t) = \tilde{K}(t)\cdot exp(\gamma_K t)$, where $\gamma_K = \gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - \rho$, $K(t)$ is not necessarily monotonic neither:

**Proposition 15.**

**(1)** *$\tilde{K}(t)$ is not necessarily monotonic.*

**(2)** *If $S(0)/K(0) = (S(0)/K(0))^*$, then $\tilde{K}(t) = K(0)$ for all $t$.*

**(3)** *$K(t)$ is not necessarily monotonic.*

**(4)** *If $S(0)/K(0) = (S(0)/K(0))^*$, then $K(t)$ increases monotonically for all $t$.*

See Appendix H, Figure 5, for an illustration.

Let us study the monotonicity of investment. Since $\tilde{I}(t) = \overline{A}\tilde{K}(t) - \tilde{C}(t)$, then $d\tilde{I}/dt = \overline{A}d\tilde{K}(t)/dt - d\tilde{C}(t)/dt$. Hence, from Sections 6.3 and 6.4, we conclude that the detrended investment is not necessarily monotonic[22]. Moreover, since $I(t) = \tilde{I}(t)\cdot exp(\gamma_I t)$, where $\gamma_I = \gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho$, the investment is not necessarily monotonic neither:

**Proposition 16.**

[22] We achieve the same conclusion by differentiating equation (40) with respect to $t$.

**(1)** *$\tilde{I}(t)$ is not necessarily monotonic.*

**(2)** *If $S(0)/K(0) = (S(0)/K(0))^*$, then*

$$\tilde{I}(t) = \overline{A}K(0)\left(1 - \alpha\frac{Z+\rho}{Z}\right)$$

*for all $t$.*

**(3)** *$I(t)$ is not necessarily monotonic.*

**(4)** *If $S(0)/K(0) = (S(0)/K(0))^*$, then $I(t)$ increases monotonically for all $t$.*

See Appendix H, Figure 6, for an illustration.

Since the detrended output $\tilde{Y}(t)$ is equal to $\overline{A}\tilde{K}(t)$, then $d\tilde{Y}(t)/dt = \overline{A}d\tilde{K}(t)/dt$. From section Proposition 15, we know that $\tilde{K}(t)$ is not necessarily monotonic, but we can not establish a general condition for this non-monotonicity. Hence, we can only conclude that $\tilde{Y}$ is not necessarily monotonic[23]. Since $Y(t) = \tilde{Y}(t) \cdot exp(\gamma_Y t)$, where $\gamma_Y = \gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho$, we conclude that $Y(t)$ is not necessarily monotonic neither:

**Proposition 17.**

**(1)** *$\tilde{Y}(t)$ is not necessarily monotonic.*

**(2)** *If $S(0)/K(0) = (S(0)/K(0))^*$, then $\tilde{Y}(t) = \overline{A}K(0)$ for all $t$.*

**(3)** *$Y(t)$ is not necessarily monotonic.*

**(4)** *If $S(0)/K(0) = (S(0)/K(0))^*$, then $Y(t)$ increases monotonically for all $t$.*

See Appendix H, Figure 7, for an illustration.

## 6.5 Monotonicity of extraction and stock of non-renewable energy resources

Let us conclude Section 6 studying the monotonicity properties of the extraction ($R(t)$) and stock ($S(t)$) of non-renewable energy resources. From

[23] As for investment, we get the same conclusion by differentiating equation (41) with respect to $t$.

Equation (42), we obtain the detrended $R(t)$:

$$\tilde{R}(t) =$$

$$\cdot \left( \frac{Z}{\varphi Z + \alpha \overline{A} \theta^{\alpha}} \right)^{\frac{1}{\alpha}} \frac{\overline{A}K(0)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha}}\right)} \left[ {}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha} exp(Zt)}\right) - \alpha \frac{Z+r}{Z} \right] \quad (51)$$

Taking logs and differentiating with respect to $t$, it yields

$$\frac{d\tilde{R}(t)}{dt} = \left[ \tilde{R}(t) \frac{Z+\rho}{2Z+\rho} \frac{Z^2}{\alpha \overline{A}\theta^{\alpha} exp(Zt)} \frac{{}_2F_1\left(a,b,c;-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha} exp(Zt)}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A}\theta^{\alpha} exp(Zt)}\right) - \alpha \frac{Z+\rho}{Z}} \right] \varphi, \quad (52)$$

where

$$a = 2; b = 1 + \frac{Z+\rho}{Z}; c = b+1.$$

Following Corollary 8, we observe that the terms between square brackets in equation (52) are positive. Then, the sign of $d\tilde{R}(t)/dt$ is determined by the sigh of $\varphi$. Hence, we establish the following proposition:

**Proposition 18.**

**(1)** *If $S(0)/K(0) > (S(0)/K(0))^{*}$, then $\tilde{R}(t)$ decreases monotonically for all $t$.*

**(2)** *If $S(0)/K(0) = (S(0)/K(0))^{*}$, then*[24]

$$\tilde{R}(t) = \overline{A}K(0) \left( \frac{Z}{\alpha \overline{A}\theta^{\alpha}} \right)^{\frac{1}{\alpha}} \left( 1 - \alpha \frac{Z+\rho}{Z} \right).$$

**(3)** *If $S(0)/K(0) < (S(0)/K(0))^{*}$, then $\tilde{R}(t)$ increases monotonically for all $t$.*

See Appendix H, Figure 8, for an illustration.

Since $R(t) = \tilde{R}(t) \cdot exp(-\rho t)$, then $dR(t)/dt = exp(-\rho t)(d\tilde{R}(t)/dt - \rho \tilde{R}(t))$. Hence, applying equation (52), we obtain

$$\frac{dR(t)}{dt} = \tilde{R}(t) exp(-\rho t)$$

[24] Notice that condition (32) implies that $1 > \alpha \frac{Z+\rho}{Z}$.

$$\cdot \left\{ \left[ \frac{Z+\rho}{2Z+\rho} \frac{Z^2}{\alpha \overline{A\theta}^{\alpha} exp(Zt)} \frac{{}_2F_1\left(a,b,c;-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} exp(Zt)}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} exp(Zt)}\right) - \alpha \frac{Z+\rho}{Z}} \right] \varphi - \rho \right\}. \quad (53)$$

Since the terms between square brackets are positives, we obtain the monotonicity properties of $R(t)$:

**Proposition 19.**

**(1)** *If $S(0)/K(0) \geq (S(0)/K(0))^*$, then $R(t)$ decreases monotonically for all $t$.*

**(2)** *If $S(0)/K(0) < (S(0)/K(0))^*$, then $R(t)$ is not necessarily monotonic.*

See Appendix H, Figure 9, for an illustration. As Proposition 19 establishes, if the proportion of non-renewable energy resources with respect to physical capital is too low, the extraction of non-renewable energy resources is not necessarily monotonic. Indeed, it is possible to generate an example of non-monotonic behavior of $R(t)$ (see Figure 11). But unfortunately, we can not establish an analytical condition for this case. Figure 10 shows a non-monotonic case of the extraction of non-renewable resources, where $R(t)$ increases during an initial period. And then, $R(t)$ decreases monotonically until zero. The economic interpretation of this case is the following. As we observe in Section 2.4, the Hotelling rule is the efficient condition for using up the non-renewable resources. Indeed, equation (8) is the Hotelling rule for our economy. For this particular case, Figure 10 shows that $I(t)$ always increases. Then, for a fixed $R(t)$, the RHS of equation (8) (*i.e.,* the growth rate of the marginal productivity of extraction of non-renewable energy resources) rises. Since the RHS of equation (8) (*i.e.,* the marginal productivity of investment) equals the LHS, then LHS has to increase too. Hence, $R(t)$ should rise, increasing the LHS and reducing the RHS. However, technical progress increases with $t$. Then, after some periods, technical progress can increase the LHS even if $R(t)$ decreases[25]. Finally, we point out that this result contrasts with Dasgupta and Heal (1974). These authors obtained that $R(t)$ decreases monotonically for all $t$. However, our model allows for a non-monotonic behavior of $R(t)$. Indeed, this possibility rises the importance of considering physical capital production as more energy-intensive than consumption sector, and the role of technical progress (in particular,

[25]Notice that energy-saving technical progress also affects the RHS, through its growth rate $\gamma_\theta$. However, we assumed that $\gamma_\theta$ is constant. Then, the effect of energy-saving technical progress on the RHS is constant with $t$. Nevertheless, the effect of technical progress (in particular, energy-saving technical progress) on the LHS increases with $t$.

energy-saving technical progress).

To conclude, we study the monotonic properties of the stock of non-renewable energy resources. Equation (10) establishes that $\dot{S}(t) = -R(t)$. Then, $dS(t)/dt = -R(t)$. Hence, since $R(t) > 0$ for all $t$, $dS(t)/dt < 0$ for all $t$:

**Proposition 20.** *$S(t)$ decreases monotonically for all $t$.*

See Appendix H, Figure 11, for an illustration.

# 7 Concluding remarks

In this paper, we studied the implications of assuming that physical capital accumulation is relatively more energy-intensive than consumption. We pointed out four main contributions of our paper. First, we concluded that technical progress (in particular, energy-saving technical progress) is a key factor to sustain long-run growth when physical capital accumulation is more energy-intensive than consumption. As we noticed in our paper, this result contrasts with Stiglitz (1974) and Dasgupta and Heal (1979), where the economy achieves steady state even without technical progress. Indeed, we showed that, in our model, neither BGP nor steady state are possible if there is not technical progress. The second contribution was the application of Special Functions representation to provide a close-form solution of the optimal solution paths of our variables in levels for every time $t$. This new technique in Economic Theory allows us to have a full characterization of both short and long-run dynamics. Indeed, the third and fourth contributions emerge thanks to this technique. Taking the close-form solutions, we provided the necessary and sufficient conditions for existence and uniqueness of BGP, which was the third contribution of our paper. Moreover, we also proved that there is asymptotic convergence to the BGP. Finally, the fourth contribution was the study of the monotonicity properties of the optimal solution paths of our variables in levels. As we pointed out, this kind of studies are typically unallowed in standard approaches. We found that our assumptions on physical capital accumulation (relatively more energy-intensive than consumption), and technical progress, implies new non-monotonic behavior of the optimal trajectories. In particular, we showed that consumption and extraction of non-renewable energy resources can depict non-monotonic behavior. Moreover, we noticed that this non-monotonic behavior of consumption and extraction contrast with the non-monotonicity result of Dasgupta and Heal (1974).

This model has several limitations. Among of them, we point out the following three limits of our analysis. First, following Sollow (1974a), we assumed unlimited (exogenous) technical progress. Regarding to our disembodied technical progress ($A(t)$), this assumption models the idea of unlimited growth of knowledge. However, unlimited energy-saving technical progress ($\theta(t)$) could arise physical incompatibility problems related to the thermodynamic laws. In the literature of exogenous energy-saving technical progress, the assumption of unlimited growth of energy efficiency is widely used (see for instance Boucekkine and Pommeret (2004), Azomahou *et al.* (2004), and Pérez-Barahona and Zou (2006a,b)). Indeed, these authors assume that the limit of improving energy efficiency is very far away. In our model, we take that point of view. However, from Section 5, one should observe that the effect of technical progress on our economy is a combination of both disembodied and energy-saving technical progress (see constant $Z$ and BGP growth rates). Then, for high enough growth rates of disembodied technical progress ($\gamma_A$), it is possible to have positive BGP without growth of energy-saving technical progress (see Proposition 2). Nevertheless, if $\gamma_A$ is not high enough, the economy could achieve positive BGP by adopting renewable energy resources. This case is not analyzed here, but it is a very interesting extension to our model since it could rise additional questions such as the depletion of non-renewable energy resources in a finite period.

A second limitation of our analysis is that we assumed exogenous technical progress. The reason of taking that assumption is the following. The aim of this paper was to study the conditions under which technical progress can offset the trade-off between economic growth and the usage of non-renewable energy resources. Then, we did not analyze how the economy can achieve these conditions. However, a very interesting extension to our model is to assume technical progress (in particular, energy-saving technical progress) as an endogenous variable. Indeed, as we observed in Proposition 2, our BGP growth rates are not affected by the stock of non-renewable energy resources. However, the scarcity of energy resources could rise new investments in energy-saving technologies, establishing an effect of the stock of non-renewable energy resource on the BGP growth rates. As it is suggested in Pérez-Barahona (2006c), a possibility to endogenize energy-saving technical progress could be by means of an R&D sector (for energy-saving technologies) following Aghion and Howitt (1992) and Grossman and Helpman (1991a,b).

Finally, as third limitation, we notice that it was assumed no physical capital depreciation. Indeed, our model is based on Dasgupta and Heal (1974), Hartwick (1989) and Pérez-Barahona (2006), which do not consider capital depreciation neither. This simplification is frequently considered in models of environmental economics (see for instance Sollow (1974a,b), Pezzey and Withagen (1998), and Stokey (1998)). However, for an accurate understanding of the relationship between non-renewable energy resources and physical capital accumulation, capital depreciation should be considered.

# References

Abramowitz M. and Stegun A., 1970. Handbook of Mathematical Functions. Dover Publications, INC., New York.

Aghion P. and Howitt P., 1992. A Model of Growth through Creative Destruction. Econometrica 60(2): 323-351.

Auty R.M., 1993. Sustaining Development in Mineral Economies: The Resource Curse Thesis. Routledge, London.

Azomahou T., Boucekkine R. and Nguyen Van P., 2004. Energy consumption, technological progress and economic policy. Mimeo. June.

Azomahou T., Boucekkine R. and Nguyen Van P., 2006. Energy consumption and vintage effect: a sectoral analysis. Mimeo.

Boucekkine R., Pommeret A., 2004. Energy saving technical progress and optimal capital stock: the role of embodiment. Economic Modelling 21, 429-444.

Boucekkine R. and Ruiz-Tamarit J.R., 2006. Special functions for the study of economic dynamics: The case of the Lucas-Uzawa model. Journal of Mathematical Economics. Forthcoming.

Carraro C., Gerlagh R. and van der Zwaan B., 2003. Endogenous technical change in environmental macroeconomics. Resource and Energy Economics 25, 1-10.

COM, 2005. Doing more with less. Green Papers. Official Documents. European Union. 265, June.

Dasgupta P.S. and Heal G.M., 1974. The Optimal Depletion of Exhaustible Resources. Review of Economic Studies, (Special Number), 3-28.

Dasgupta P.S. and Heal G.M., 1979. Economic Theory and Exhaustible Resources. James Nisbet and Cambridge University Press, 1979, pp.1-501.

Grossman G.H. and Helpman E., 1991a. Quality Ladders in the Theory of Growth. Review of Economic Studies, vol. 58, no. 1, January.

Grossman G.H. and Helpman E., 1991b. Quality Ladders and Product Cycles. Quarterly journal of Economics, vol. 106, no. 425, May.

Hartwick J.M., 1989. Non- renewable Resources: Extraction Programs and Markets. London: Harwood Academic.

Löschel A., 2002. Technological change in economic models of environmental policy: a Survey. Ecological Economics 43, 105-126.

Luke L.Y., 1969. The Special Functions and their approximations. Volume 1 and 2. Academic Press, Inc. New York.

Newell R., Jaffe A. and Stavins R., 1999. The Induced Innovation Hypothesis and Energy-saving Technological Change. Quarterly Journal of Economics 114, 941-975.

Pérez-Barahona A. and Zou B., 2006a. A comparative study of energy saving technical progress in a vintage capital model. Resource and Energy Economics, Volume 28, Issue 2, May 2006, 181-191.

Pérez-Barahona A. and Zou B., 2006b. Energy saving technological progress in a vintage capital model. Economic Modelling of Climate Change and Energy Policies. Edward Elgar, 166-179.

Pérez-Barahona, 2006c. The problem of non-renewable energy resource in the production of physical capital. CORE Discussion Papers. Forthcoming.

Pezzey J. and Withagen C.A., 1998. The Rise, Fall and Sustainability of Capital-Resource Economics. Scandinavian Journal of Economics. 100(2), 513-527.

Sach J.D., Warner M., 1995. Natural resource abundance and economic growth. NBER Working Paper. 5398.

Smulders S., Michiel de Nooij, 2003. The impact of energy conservation on technology and economic growth. Resource and Energy Economics 25, 59-79.

Solow, Robert M, 1974a. The Economics of Resources or the Resources of Economics, American Economic Review. American Economic Association, vol. 64(2), 1-14.

Solow, Robert M, 1974b. Intergenerational Equity and Exhaustible Resource. Review of Economic Studies, (Special Number), 29-46.

Stiglitz J., 1974. Growth with Exhaustible Natural Resources: Efficient and Optimal Growth Paths. Review of Economic Studies, (Special Number), 123-137.

Stokey N.L., 1998. Are there limits to growth?. International Economic Review. February. Vol. 39, No. 1.

Sydsæter K., Strøm A. and Berck P, 2000. Economists' Mathematical Manual, Spinger.

# Appendix A

Following Sydsæter *et al.* (1999, pages 109-110), the Lagrangian associated with our optimal control problem with mixed constraints is

$$\mathcal{L}(\bullet) = U[A(t) - I(t)]exp(-\rho t) + \psi(t)\{g[\theta(t), I(t), R(t)]\} - \lambda(t)R(t) + q(t)S(t),$$

where $\psi(t)$, $\lambda(t)$ and $q(t)$ are the Lagrangian multipliers, $U[\bullet]$ is a general utility function, and $g[\bullet]$ is a general production function for physical capital accumulation.

The corresponding first order conditions (FOC) are:

$$\frac{\partial \mathcal{L}}{\partial I(t)} = 0; \frac{\partial \mathcal{L}}{\partial R(t)} = 0;$$

$$\frac{\partial \mathcal{L}}{\partial K(t)} = -\dot{\psi}(t); \frac{\partial \mathcal{L}}{\partial S(t)} = -\dot{\lambda}(t);$$

$$q(t) \geq 0 (= 0 \text{ if } S(t) > 0);$$

$$\lim_{t \to \infty} \psi(t)K(t) = 0 \text{ and} \lim_{t \to \infty} \lambda(t)S(t) = 0 \text{ (Transvarsality conditions).}$$

From the FOC, one obtains

$$\psi(t)g_I = U_c exp(-\rho t), \tag{A.1}$$

$$\psi(t)g_R = \lambda(t) + q(t), \tag{A.2}$$

$$\dot{\psi}(t) = -U_c A(t) exp(-\rho t), \tag{A.3}$$

$$q(t) = -\dot{\lambda}(t). \tag{A.4}$$

Since it is not optimal to completely deplete the stock of non-renewable energy resource in a finite $t$[26], *i.e.*, $S(t) > 0$ for all $t$, then $q(t) = 0$. Taking equation (A.4), this implies $\lambda(t) = \lambda$ for all $t$.

Applying equation (A.1) into equation (A.3), one gets

$$\frac{\dot{\psi}(t)}{\psi(t)} = -A(t)g_I. \tag{A.5}$$

Taking logs in equation (A.2) and applying equation (A.5), one obtains the Hotelling rule

$$A(t)g_I = \frac{\dot{g}_R}{g_R}.$$

Taking the functional forms of my model, equation (8) is easily obtained from the Hotelling rule.

Taking logs in equation (A.1) and applying equation (A.5), one finds the Ramsey (saving) rule

$$\frac{U_{cc}\dot{C}}{U_c} - \rho = \frac{\dot{g}_I}{g_I} - Ag_I.$$

Applying the functional forms of my model, one gets equation (7).

Finally, equation (6) is the technology for physical capital accumulation, equation (9) is the budget constraint of the economy, and equation (10) is the law of motion of the stock of non-renewable energy resource. ■

# Appendix B

Along the BGP, all the endogenous variables grow at constant rate, *i.e.*, $x(t) = \overline{x} \cdot exp(\gamma_x t)$. Moreover, we assume that $A(t) = \overline{A} \cdot exp(\gamma_A t)$ and $\theta(t) =$

[26] Notice that non-renewable energy resources is an essential input for physical capital accumulation. Moreover, energy-saving technical progress is embodied in the new equipment goods.

$\overline{\theta} \cdot exp(\gamma_\theta t)$ for all $t$, where $\overline{A}, \overline{\theta}, \gamma_A, \gamma_\theta$ are strictly positive and exogenous parameters.

By writing equation (7) along the BGP, one obtains

$$\gamma_A + \alpha\gamma_\theta = \alpha(\gamma_I - \gamma_R), \qquad (B.1)$$

which implies that

$$\gamma_C = (1-\alpha)\overline{A\theta}^\alpha \left(\frac{\overline{R}}{\overline{I}}\right)^\alpha + \gamma_A - \rho. \qquad (B.2)$$

Since $Y(t) = C(t) + I(t)$, $\gamma_Y = \gamma_I = \gamma_C$. Moreover, from the final good technology ($Y(t) = A(t)K(t)$) one gets that $\gamma_K = \gamma_Y - \gamma_A$. Then,

$$\gamma_K = (1-\alpha)\overline{A\theta}^\alpha \left(\frac{\overline{R}}{\overline{I}}\right)^\alpha. \qquad (B.3)$$

From equation (8) along the BGP, it is easy to obtain that

$$\left(\frac{\overline{R}}{\overline{I}}\right)^\alpha = \frac{\alpha\gamma_\theta + (1-\alpha)(\gamma_I - \gamma_R)}{(1-\alpha)\overline{A\theta}^\alpha}. \qquad (B.4)$$

Applying equation (B.4) into equation (B.3), one gets the value of the ratio

$$\left(\frac{\overline{R}}{\overline{I}}\right)^\alpha = \frac{\gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A}{(1-\alpha)\overline{A\theta}^\alpha}. \qquad (B.5)$$

Replacing this ratio into equations (B.2) and (B.3), one obtains

$$\gamma_Y = \gamma_C = \gamma_I = \gamma_\theta + \frac{1}{\alpha}\gamma_A - \rho$$

and

$$\gamma_K = \gamma_\theta + \frac{1-\alpha}{\alpha}\gamma_A - \rho.$$

From equation (B.1) and the two previous expressions, one finds that the long-run growth rate of the energy is $\gamma_R = -\rho$. Finally, since $\dot{S}(t) = -R(t)$,

$$\gamma_S = \gamma_R (= -\rho) \blacksquare$$

# Appendix C

Equation (21) can be rewritten as a standard linear first-order differential equation:

$$\dot{\tilde{K}}(t) + a(t)\tilde{K}(t) = b(t), \tag{C.1}$$

where

$$a(t) = \frac{1-\alpha}{\alpha}Z - \rho - \overline{A\theta}^{\alpha} X(t)^{\alpha},$$
$$b(t) = -\overline{\theta}^{\alpha} X(t)^{\alpha}\tilde{C}(t),$$

and $K(0)(= \tilde{K}(0))$ is given.

Since the previous steps provide the solutions for $X(t)$ (equation (18)) and $\tilde{C}(t)$ (equation (20)), one can solve the differential equation (C.1) applying the following result (Sydsæter *et al.* (1999)):

$$\tilde{K}(t) = \tilde{K}(0)exp\left(-\int_0^t a(\xi)d\xi\right) + \int_0^t b(\tau)exp\left(-\int_\tau^t a(\xi)d\xi\right)d\tau. \tag{C.2}$$

Taking equations (18) and (20), one obtains that

$$a(t) = \frac{1-\alpha}{\alpha}Z - \rho - \frac{Z\overline{A\theta}^{\alpha}}{\frac{\varphi Z}{exp(Zt)} + \alpha\overline{A\theta}^{\alpha}}. \tag{C.3}$$

After some calculations, one finds

$$\int a(\xi)d\xi = \left(\frac{1-\alpha}{\alpha}Z - \rho\right)\xi + \frac{1}{\alpha}ln\left(\frac{\varphi Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}exp(Z\xi)}\right). \tag{C.4}$$

Evaluating the integral (C.4) in $t$ and 0, and after rearranging terms, one obtains the first part of equation (C.2):

$$\tilde{K}(0)exp\left(-\int_0^t a(\xi)d\xi\right) = \tilde{K}(0)exp\left(-\frac{1}{\alpha}ln(\varphi Z + \alpha\overline{A\theta}^{\alpha})\right)$$
$$\cdot exp\left(\frac{1}{\alpha}(ln(\varphi Z + \alpha\overline{A\theta}^{\alpha}) - ((1-\alpha)Z - \alpha\rho)t)\right).$$

In order to calculate the second part of equation (C.2), we first obtain

$$b(\tau)exp\left(-\int_\tau^t a(\xi)d\xi\right) =$$

$$-Z\overline{\theta}^{\alpha}C(0)exp\left\{\frac{1}{\alpha}[ln(\varphi Z + \alpha\overline{A\theta}^{\alpha}exp(Zt)) - ln(\varphi Z + \alpha\overline{A\theta}^{\alpha})]\right\}$$

$$\cdot exp\left\{\left(-\frac{1-\alpha}{\alpha}Z+\rho\right)t\right\}\frac{exp(-\rho\tau)}{\varphi Z+\alpha\overline{A}\overline{\theta}^{\alpha}exp(Z\tau)}, \qquad (C.5)$$

by evaluating the integral (C.4) in $\tau$ and $t$. Then,

$$\int_0^t b(\tau)exp\left(-\int_\tau^t a(\xi)d\xi\right)d\tau =$$

$$-Z\overline{\theta}^{\alpha}C(0)exp\left\{\frac{1}{\alpha}[ln(\varphi Z+\alpha\overline{A}\overline{\theta}^{\alpha}exp(Zt))-ln(\varphi Z+\alpha\overline{A}\overline{\theta}^{\alpha})]\right\}$$

$$\cdot exp\left\{\left(-\frac{1-\alpha}{\alpha}Z+\rho\right)t\right\}$$

$$\cdot\int_0^t \frac{exp(-\rho\tau)}{\varphi Z+\alpha\overline{A}\overline{\theta}^{\alpha}exp(Z\tau)}d\tau. \qquad (C.6)$$

In order to calculate equation (C.6), we need to obtain

$$\int_0^t \frac{exp(-\rho\tau)}{\varphi Z+\alpha\overline{A}\overline{\theta}^{\alpha}exp(Z\tau)}d\tau. \qquad (C.7)$$

To do this, we follow Boucekkine and Ruiz-Tamarit (2004). Equation (C.7) is equal to

$$\int_0^t \left(\varphi Zexp(\rho\tau)+\alpha\overline{A}\overline{\theta}^{\alpha}exp((Z+\rho)\tau)\right)^{-1}d\tau.$$

Taking the following change of variable $V=exp(-(\rho+Z)\tau)$,

$$\int_0^t \left(\varphi Zexp(\rho\tau)+\alpha\overline{A}\overline{\theta}^{\alpha}exp((Z+\rho)\tau)\right)^{-1}d\tau =$$

$$\int_1^{exp(-(\rho+Z)t)}\left(1+\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}V^{\frac{Z}{Z+\rho}}\right)^{-1}dV. \qquad (C.8)$$

Recalling $y=exp(-(\rho+Z)t)$, $A_0=\frac{\varphi Z}{\alpha\overline{A}\overline{\theta}^{\alpha}}$, and $\beta=-1$, equation (C.8) can be rewritten as

$$\int_1^y \left(1+A_0V^{\frac{Z}{Z+\rho}}\right)^{\beta}dV. \qquad (C.9)$$

Applying the binomial theorem:

$$(1+ax^b)^c=\sum_{n=0}^{\infty}(-c)_n\frac{(-ax^b)^n}{n!},$$

equation (C.9) equals

$$\sum_{n=0}^{\infty} \frac{(-\beta)_n}{n!}(-A_0)^n \int_1^y V^{\frac{Z}{Z+\rho}n} dV,$$

which is equal to

$$\sum_{n=0}^{\infty}(-\beta)_n \frac{-(A_0)^n}{n!} \frac{1}{\frac{Z}{Z+\rho}n+1}\left(y^{\frac{Z}{Z+\rho}n}y-1\right). \qquad (C.10)$$

Applying the property $\frac{X}{X+n} = \frac{(X)_n}{(1+X)_n}$, equation (C.10) equals

$$y\sum_{n=0}^{\infty} \frac{(a)_n(-\beta)_n}{(1+a)_n} \frac{(-A_0 y^{\frac{Z}{Z+\rho}})^n}{n!}$$

$$-\sum_{n=0}^{\infty} \frac{(a)_n(-\beta)_n}{(1+a)_n} \frac{(-A_0)^n}{n!}. \qquad (C.11)$$

Taking the definition of Gauss hypergeometric function

$${}_2F_1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!},$$

one finds that equation (C.8) equals

$$exp(-(\rho+Z)t)\,{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} exp(Zt)}\right) - {}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right), \qquad (C.12)$$

where

$${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} exp(Zt)}\right) = {}_2F_1\left(1, \frac{Z+\rho}{Z}, 1+\frac{Z+\rho}{Z}; -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} exp(Zt)}\right),$$

$${}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right) = {}_2F_1\left(1, \frac{Z+\rho}{Z}, 1+\frac{Z+\rho}{Z}; -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right)$$

Hence, retrieving the change of variables,

$$\int_0^t \frac{exp(-\rho\tau)}{\varphi Z + \alpha \overline{A\theta}^{\alpha} exp(Z\tau)} d\tau =$$

$$\frac{1}{\alpha \overline{A\theta}^{\alpha}(Z+\rho)}\left[{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}}\right) - \frac{1}{exp[(\rho+Z)t]}\,{}_2F_1\left(-\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} exp(Zt)}\right)\right]. \qquad (C.13)$$

Finally, taking together equations (C.3), (C.5), and (C.13), and rearranging terms, one can easily obtain equation (22) ■

# Appendix D

Equation (24) can be rewritten as follows:

$$\tilde{R}(t) = [a] + [b] + [c] + [d],$$

where

$$[a] = (\varphi Z + \alpha \overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} \overline{A} K(0) exp\left\{ \left( \frac{2-\alpha}{\alpha} Z - \rho \right) t \right\},$$

$$[b] = (\varphi Z + \alpha \overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}} \frac{Z^{\frac{1+\alpha}{\alpha}}}{Z+\rho} \frac{C(0)}{\alpha} {}_2F_1\left( -\frac{\varphi Z}{\alpha \overline{A}\overline{\theta}^{\alpha} exp(Zt)} \right) exp\left\{ 2\left( \frac{1-\alpha}{\alpha} Z - \rho \right) t \right\},$$

$$[c] = -(\varphi Z + \alpha \overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}} \frac{Z^{\frac{1+\alpha}{\alpha}}}{Z+\rho} \frac{C(0)}{\alpha} {}_2F_1\left( -\frac{\varphi Z}{\alpha \overline{A}\overline{\theta}^{\alpha}} \right) exp\left\{ \left( \frac{2-\alpha}{\alpha} Z - r \right) t \right\},$$

$$[d] = -(\varphi Z + \alpha \overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} C(0).$$

Then,

$$\int_0^t \tilde{R}(\tau) exp(-\rho\tau) d\tau = \int_0^t [a] exp(-\rho\tau) d\tau + \int_0^t [b] exp(-\rho\tau) d\tau$$

$$+ \int_0^t [c] exp(-\rho\tau) d\tau + \int_0^t [d] exp(-\rho\tau) d\tau.$$

It is easy to prove that

$$\int_0^t [a] exp(-\rho\tau) d\tau =$$

$$\frac{(\varphi Z + \alpha \overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} \overline{A} K(0)}{2(\frac{1}{\alpha} Z - \rho) - Z} \left\{ exp\left[ \left( 2\left( \frac{1}{\alpha} Z - \rho \right) - Z \right) t \right] - 1 \right\} = [1]$$

$$\int_0^t [c] exp(-\rho\tau) d\tau = -A_2 \frac{{}_2F_1\left( -\frac{\varphi Z}{\alpha \overline{A}\overline{\theta}^{\alpha}} \right)}{2(\frac{1}{\alpha} Z - \rho) - Z} \left\{ exp\left[ \left( 2\left( \frac{1}{\alpha} Z - \rho \right) - Z \right) t \right] - 1 \right\} = [3]$$

$$\int_0^t [d] exp(-\rho\tau) d\tau = \frac{(\varphi Z + \alpha \overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}} Z^{\frac{1}{\alpha}} C(0)}{\rho} (exp(-\rho t) - 1) = [4],$$

where

$$A_2 = (\varphi Z + \alpha \overline{A}\overline{\theta}^{\alpha})^{-\frac{1}{\alpha}} \frac{Z^{\frac{1+\alpha}{\alpha}}}{Z+\rho} \frac{C(0)}{\alpha}.$$

The integral [b] involves ${}_3F_2$ Hypergeometric functions. Indeed, applying the Euler integral representation (Abramowitz and Stegun (1970), page 558, formula 15.3.1), we obtain that

$$\int_0^t [b] exp(-\rho\tau) d\tau = A_2 \frac{Z+\rho}{Z} (\alpha \overline{A}\overline{\theta}^{\alpha})^{\frac{Z+\rho}{Z}}$$

$$\cdot \int_0^1 (1-\xi)^{\frac{\rho}{Z}} \left( \int_0^t \left( \alpha\overline{A\theta}^{\alpha} exp(Z\tau) + \varphi Z\xi \right)^{-\frac{Z+\rho}{Z}} exp\left\{ \left( \frac{2-\alpha}{\alpha} Z - 2\rho \right) \tau \right\} d\tau \right) d\xi,$$

The integral $\int_0^t (\cdot) d\tau$ can be rewritten as

$$\int_0^t \left\{ \varphi Z\xi exp(R\tau) + \alpha\overline{A\theta}^{\alpha} exp[(Z+\Omega)\tau] \right\}^{-\frac{Z+\rho}{Z}} d\tau, \qquad (D.1)$$

where

$$\Omega = -\left( \frac{2-\alpha}{\alpha} Z - 2\rho \right) \frac{Z}{Z+\rho}.$$

Following a similar strategy as in Appendix C (equation (C.7)), we can apply the change of variable $V = exp\{(\Omega+\gamma)\tau\}$ to equation (D.1). Taking $\gamma = -(Z+\Omega)[(Z+\rho)/Z] - \Omega$, one can prove that equation (D.1) equals

$$-\left( \alpha\overline{A\theta}^{\alpha} \right)^{-\frac{Z+\rho}{Z}} \frac{Z}{Z+\rho} \frac{1}{Z+\Omega}$$

$$\cdot \left[ exp\left\{ -(Z+\Omega)\frac{Z+\rho}{Z} t \right\} {}_2F_1 \left( \hat{a}, \hat{b}, \hat{c}; -\frac{\varphi Z\xi}{\alpha\overline{A\theta}^{\alpha} exp(Zt)} \right) - {}_2F_1 \left( \hat{a}, \hat{b}, \hat{c}; -\frac{\varphi Z\xi}{\alpha\overline{A\theta}^{\alpha}} \right) \right],$$

where

$$\hat{a} = \frac{Z+\rho}{Z}\frac{Z+\Omega}{Z}; \hat{b} = \frac{Z+\rho}{Z}; \hat{c} = 1 + \frac{Z+\rho}{Z}\frac{Z+\Omega}{Z}.$$

Then,

$$\int_0^t [b] exp(-\rho\tau) d\tau = A_2 \frac{1}{Z+\Omega}$$

$$\cdot \left\{ [INT1] - exp\left( -(Z+\Omega)\frac{Z+\rho}{Z} t \right) [INT2] \right\}, \qquad (D.2)$$

where

$$[INT1] = \int_0^1 (1-\xi)^{\frac{\rho}{Z}} {}_2F_1 \left( \hat{a}, \hat{b}, \hat{c}; -\frac{\varphi Z\xi}{\alpha\overline{A\theta}^{\alpha}} \right) d\xi,$$

$$[INT2] = \int_0^1 (1-\xi)^{\frac{\rho}{Z}} {}_2F_1 \left( \hat{a}, \hat{b}, \hat{c}; -\frac{\varphi Z\xi}{\alpha\overline{A\theta}^{\alpha} exp(Zt)} \right) d\xi.$$

Assuming that $\left| -\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}} \right| < 1$, we can apply the following property (see Luke (1975), Vol.1, page 58, equation (10)):

$$_{p+1}F_{q+1} \left\{ (\beta, \alpha_p); (\beta+\sigma, \rho_p); Z \right\} =$$

$$\frac{\Gamma(\beta+\sigma)}{\Gamma(\beta)\Gamma(\sigma)} \int_0^1 t^{\beta-1} (1-t)^{\sigma-1} {}_{p+1}F_{q+1} \left\{ (\alpha_p); (\rho_p\ Zt) \right\} dt$$

for $|Z| < 1$, $Re(\beta) > 0$, and $Re(\sigma) > 0$.

In our case, $t \equiv \xi$, $p \equiv 2$, $q \equiv 1$, $\beta \equiv 1$, $\sigma \equiv \frac{r}{Z} + 1$,

$$\alpha_p \equiv \left( \frac{Z+\rho}{Z} \frac{Z+\Omega}{Z}, \frac{Z+\rho}{Z} \right),$$

$$\rho_p \equiv \left( 1 + \frac{Z+\rho}{Z} \frac{Z+\Omega}{Z} \right),$$

and $Z \equiv -\frac{\varphi Z \xi}{\alpha \overline{A\theta}^{\alpha}}$ for [INT1], and $Z \equiv -\frac{\varphi Z \xi}{\alpha \overline{A\theta}^{\alpha} exp(Zt)}$ for [INT2]. Then,

$$[INT1] = \frac{Z}{Z+\rho} {}_3F_2 \left( a; b; -\frac{\varphi Z \xi}{\alpha \overline{A\theta}^{\alpha}} \right),$$

$$[INT1] = \frac{Z}{Z+\rho} {}_3F_2 \left( a; b; -\frac{\varphi Z \xi}{\alpha \overline{A\theta}^{\alpha} exp(Zt)} \right),$$

where

$$a = \left( 1, \frac{Z+\rho}{Z}, \frac{Z+\rho}{Z} \frac{Z+\Omega}{Z} \right),$$

$$b = \left( 1 + \frac{Z+\rho}{Z}, 1 + \frac{Z+\rho}{Z} \frac{Z+\Omega}{Z} \right).$$

Applying this result into equation (D.2), we obtain [2] ■

# Appendix E

Under the conditions (32), (36), and ${}_2F_1 \left( -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}} \right) > 0$, it is clear that the only problem is the term

$$\frac{{}_2F_1 \left( -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} exp(Zt)} \right)}{{}_2F_1 \left( -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}} \right)}$$

in equation (39). Since we know that the denominator is positive, we have to verify that the numerator is positive too. To do that, we use the following property (see Abramowitz and Stegun, 15.2.1, page 557):

$$\frac{d}{dz} {}_2F_1(a, b, c; z) = \frac{ab}{c} {}_2F_1(a+1, b+1, c+1; z) \qquad (E.1).$$

Since our $a, b, c$ terms and ${}_2F_1(a, b, c; z)$ are positive, we conclude that ${}_2F_1(a+1, b+1, c+1; z)$ and the derivative are positive too. Then, ${}_2F_1(a, b, c; z)$ increases as $z$ raises. Since $exp(Zt)$ increases our $z$ term, then

$$ {}_2F_1 \left( -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha} exp(Zt)} \right) > {}_2F_1 \left( -\frac{\varphi Z}{\alpha \overline{A\theta}^{\alpha}} \right) > 0$$

■

# Appendix F

Let us define

$$F\left(\frac{S(0)}{K(0)}\right)=\frac{S(0)}{K(0)}-\alpha\overline{A}\left(\frac{Z}{\varphi Z+\alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}$$

$$\cdot\left[\frac{Z}{Z+\rho}\frac{1}{\alpha}\frac{1}{Z+R}\frac{{}_3F_2\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}-\frac{Z+\rho}{Z}\frac{1}{\rho}\frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}\right]. \qquad (F.1)$$

Condition (35) implies

$$F\left(\frac{S(0)}{K(0)}\right)=0.$$

Then, applying the implicit function theorem, one obtains

$$\frac{\partial\varphi}{\partial(S(0)/K(0))}=-\frac{\partial F/\partial(S(0)/K(0))}{\partial F/\partial\varphi}. \qquad (F.2)$$

From (F.1), $\partial F/\partial(S(0)/K(0))=1$. Moreover, $\partial F/\partial\varphi=-\partial g(\varphi)/\partial\varphi$, where

$$g(\varphi)=\alpha\overline{A}\left(\frac{Z}{\varphi Z+\alpha\overline{A\theta}^{\alpha}}\right)^{\frac{1}{\alpha}}$$

$$\cdot\left[\frac{Z}{Z+\rho}\frac{1}{\alpha}\frac{1}{Z+R}\frac{{}_3F_2\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}-\frac{Z+\rho}{Z}\frac{1}{\rho}\frac{1}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}\right]. \qquad (F.3)$$

Applying the following property (Luke (1975), Vol.1, page 111 (Property 38))

$${}_3F_2\left\{(a,b,c);(d+1,c+1);Z\right\}=\frac{c}{c-d}{}_2F_1\left(a,b,d+1;Z\right)$$

$$-\frac{d}{c-d}{}_3F_2\left\{(a,b,c);(d,c+1);Z\right\},$$

one obtains that

$$\frac{{}_3F_2\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}=\frac{Z+\Omega}{\Omega}-\frac{Z}{\Omega}\frac{{}_2F_1\left(a',b',c',-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)}{{}_2F_1\left(-\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\right)},$$

where

$$a'=1;b'=\frac{Z+\rho}{Z}\frac{Z+\Omega}{Z};c'=1+b'.$$

Then, we can rewrite $g(\varphi)$ with ${}_2F_1$ functions. Applying the formula (E.1) for the derivative of ${}_2F_1$, one gets $\partial g(\varphi)/\partial\varphi$. Rearranging terms and taking Corollary 6, we obtain that $\partial g(\varphi)/\partial\varphi < 0$. Hence, from (F.2), one concludes that

$$\frac{\partial\varphi}{\partial(S(0)/K(0))} < 0$$

■

# Appendix G

Let us consider the case

$$\gamma_C < \frac{Z}{\alpha}\frac{\varphi Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}}.$$

Taking

$$\gamma_C = \frac{Z}{\alpha}\frac{\varphi Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha} exp(Zt^*)}, \tag{H.1}$$

we obtain $t^*$. From (H.1), one gets

$$exp(Zt^*) = \frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\left(\frac{Z - \alpha\gamma_C}{\alpha\gamma_C}\right). \tag{H.2}$$

Since, under Corollary 8, the term $Z - \alpha\gamma_C$ is strictly positive, we can apply logs in (H.2). Then,

$$t^* = \frac{ln\left[\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\left(\frac{Z-\alpha\gamma_C}{\alpha\gamma_C}\right)\right]}{Z}.$$

If the term between square brackets is $> 1$, then $t^* > 0$. We can prove it by contradiction. Let us assume that

$$\frac{\varphi Z}{\alpha\overline{A\theta}^{\alpha}}\left(\frac{Z - \alpha\gamma_C}{\alpha\gamma_C}\right) \leq 1.$$

Then, it easy to prove that

$$\gamma_C \geq \frac{Z}{\alpha}\frac{\varphi Z}{\varphi Z + \alpha\overline{A\theta}^{\alpha}},$$

which contradicts our initial statement. Then $t^* > 0$ ■

# Appendix H

Notice that this is not a numerical simulation. We proceed as follows. First, for given values of the parameters, we obtain the corresponding constant $\varphi$ by solving numerically the non-linear equation (35)[27]. Once $\varphi$ is known, we plot the optimal solution paths provided in Section 5.

## Parametrization

| $\alpha$ | $\gamma_\theta$ | $\gamma_A$ | $\overline{A}$ | $\bar{\theta}$ | $(S(0)/K(0))^*$ | $\rho$ |
|---|---|---|---|---|---|---|
| 0.32 | 0.019 | 0.01 | 1 | 2 | 0.0056 | 0.03 |

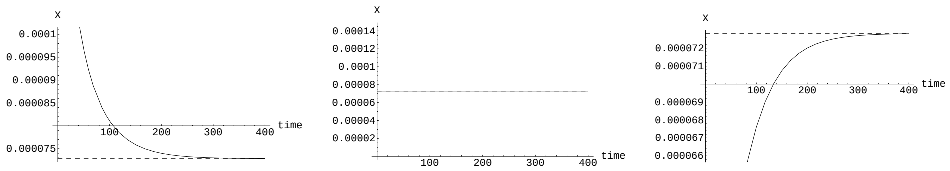
## Note for the Figures

- All Figures share the same parametrization except Figure 4, where we specify the parameters value.
- $\varphi$ is the constant corresponding to the ratio $S(0)/K(0)$. In particular, $\varphi = 0$ is the constant corresponding to the ratio $(S(0)/K(0))^*$.
- $x(t)$ represents the variable and $detrx(t)$ is the corresponding detrended variable.

## Figures

Figure 1: $X(t)$

$\varphi = -4.73$ $\qquad$ $\varphi = 0$ $\qquad$ $\varphi = 3.37$




[27]Since our problem is concave, the solution paths are unique. Then, equation (35) has a unique solution for $\varphi$. A formal proof of uniqueness of the solution of equation (35) could be done. However, this is beyond the objective of this paper. The uniqueness can be tested by plotting equation (35), for given values of the parameters.

Figure 2: $\tilde{C}(t)$

$\varphi = -4.73$ $\qquad$ $\varphi = 0$ $\qquad$ $\varphi = 3.37$

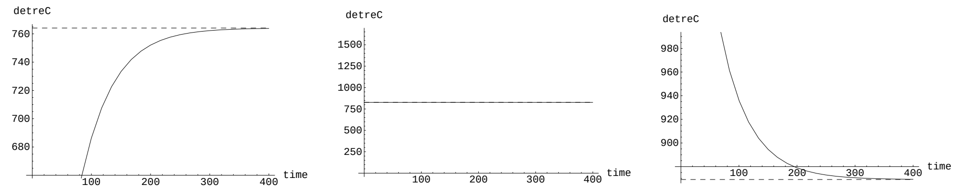



Figure 3: $C(t)$

$\varphi = -4.73$ $\qquad$ $\varphi = 0$ $\qquad$ $\varphi = 3.37$

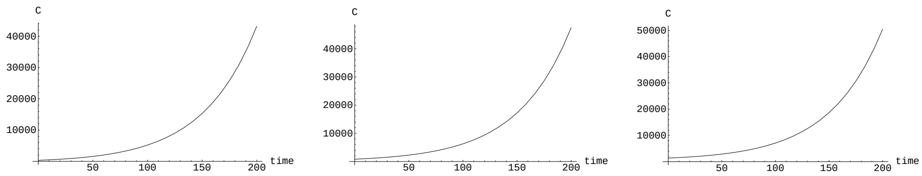



Figure 4: special case of $C(t)$

| $\alpha$ | $\gamma_\theta$ | $\gamma_A$ | $\overline{A}$ | $\overline{\theta}$ | $S(0)/K(0)$ | $\rho$ |
|---|---|---|---|---|---|---|
| 0.32 | 0.01 | 0.001 | 1 | 2 | 0.00015 | 0.01 |

$\varphi = 26.3$ $\qquad$ $\varphi = 26.3$

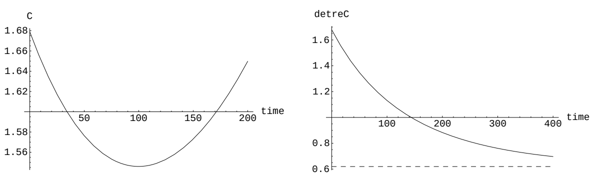


Figure 5: $\tilde{K}(t)$ and $K(t)$

$\varphi = 0$ $\qquad$ $\varphi = 0$

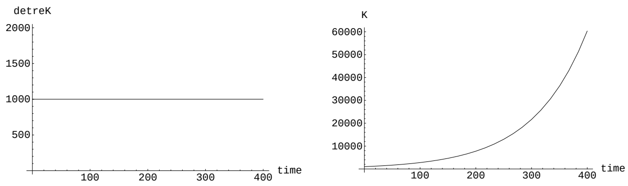



Figure 6: $\tilde{I}(t)$ and $I(t)$

$\varphi = 0$ $\qquad$ $\varphi = 0$

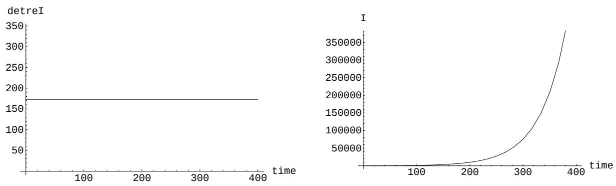



Figure 7: $\tilde{Y}(t)$ and $Y(t)$

$\varphi = 0$ $\qquad$ $\varphi = 0$

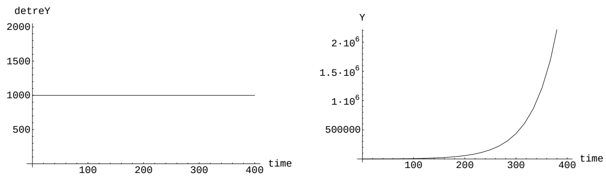



Figure 8: $\tilde{R}(t)$

$\varphi = -4.73$ $\qquad$ $\varphi = 0$ $\qquad$ $\varphi = 3.37$

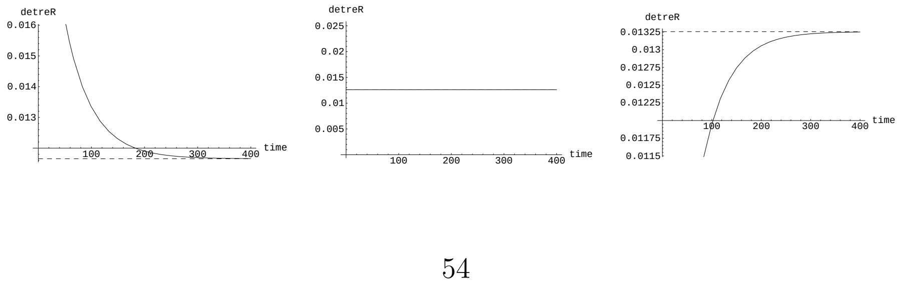


Figure 9: $R(t)$

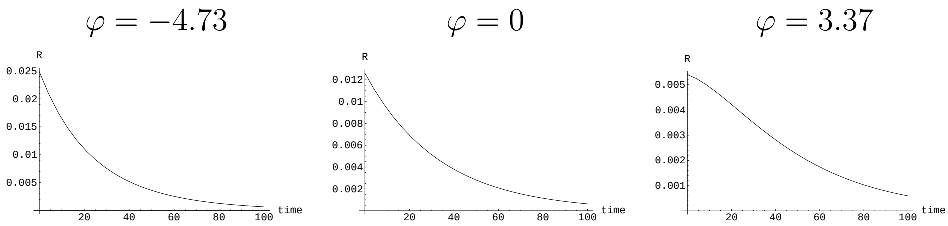



Figure 10: special case of $R(t)$

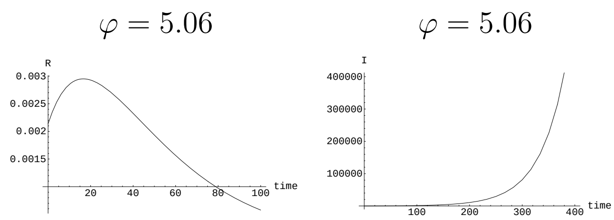



Figure 11: $S(t)$